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Randall-Sundrum models and the regularized AdS/CFT correspondence

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Abstract: It has been proposed that Randall-Sundrum models can be holographically described by a regularized (broken) conformal field theory. We analyze the foundations of this duality using a regularized version of the AdS/CFT correspondence. We compare two- and three-point correlation functions and find the same behaviour in both descriptions. In particular, we show that the regularization of the deformed CFT generates kinetic terms for the sources, which hence can be naturally treated as dynamical fields. We also discuss the counterterms required for two- and three-point correlators in the renormalized AdS/CFT correspondence.

Keywords: Renormalization Regularization and Renormalons, Extra Large Dimensions, AdS/CFT Correspondance.

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Contents

1. Introduction

The work of L. Randall and R. Sundrum has brought about a great interest in models with warped extra dimensions. With appropriate stabilization mechanisms [1, 2], some of these models can account for the large hierarchy between the Planck and the electroweak scales [3]. On the other hand, non-compact extra dimensions may be compatible with current observations when the space-time is warped [4].

In the Randall-Sundrum models and their generalizations, space-time is a piece of AdS. It is natural, hence, to try to take advantage of the AdS/CFT correspondence [5, 6, 7], which relates a quantum field theory in d dimensions to a theory in $d+1$ dimensions including gravity. J. Maldacena has suggested [8] that the non-compact Randall-Sundrum model (RS2) is dual to 4-D gravity coupled to a strongly coupled conformal theory with an ultraviolet cutoff. This idea has been realized more explicitly in [9, 10, 11] (see also [12, 13] for a string-theoretical construction), where it is further argued that 4-D gravity arises from the local counterterms that are required to define finite AdS amplitudes in the AdS/CFT correspondence. Other developements in this direction include [14, 15, 16]. This holographic description has been invoked in a number of works to study different aspects of Randall-Sundrum models (see references from [17] to [34] for an incomplete list). More recently, the duality has been extended to the Randall-Sundrum compact model (RS1) in [35, 36]. This has allowed the authors of these papers to discuss important phenomenological aspects of RS1 using the holographic description.

The purpose of this paper is to study in greater detail the basis of this duality. To start with, we shall argue that the content of the duality itself must be qualified: the regularization of the CFT generates kinetic terms for the 4-D gravity. This provides an attractive holographic picture of Randall-Sundrum models, since propagating gravity arises from regularization, just as in the AdS description. We shall support this interpretation with evidence from the calculation of correlation functions in both sides of the correspondence. Specifically, we shall be interested in correlation functions in the effective theory where all the bulk degrees of freedom have been integrated out (in a classical approximation). In order to compare with the CFT, it will be convenient to pose the gravity calculations in a way that resemble the standard AdS/CFT calculations. Our results are also relevant to the renormalized AdS/CFT correspondence [37]–[40] and we shall make some observations about the renormalization of the CFT in this context.

The layout of the paper is as follows. In section 2 we argue that RS2 is dual to a regularized CFT which automatically contains gravity. In section 3 we develop the effective formalism for a bulk scalar field in RS2 and calculate explicitly two and three-point functions. We discuss the relation to the Kaluza-Klein description and perform an expansion in the position of the Planck brane, which allows comparison with the CFT calculations. In section 4 we describe the regularization of the CFT in the AdS/CFT correspondence, and compare with the AdS calculations of the previous section. We also discuss what counterterms are required to renormalize the CFT. In section 5 we study the duality for RS1. In this case we only calculate explicitly two-points functions. In section 6 we study sources localized on the Planck brane or inside the bulk. We finish with some general comments in section 7.

2. Randall-Sundrum as a regularized CFT

There are only a few examples of the AdS/CFT correspondence in which both the gravity and the boundary theories are known. The best stablished one is the equivalence of $\mathcal{N}=4$ super Yang-Mills in 4 flat dimensions and type-IIB string theory on $AdS_5 \times S_5$, especially in the limit of strong 't Hooft coupling and large N, in which the gravity theory reduces to classical type-IIB supergravity. Other interesting cases with less number of supersymmetries and even non conformal have also been studied. More generally, it is commonly believed that any consistent gravity theory on an asymptotically AdS space has some kind of conformal dual. Here we assume that this is the case, at least for the gravity theories in Randall-Sundrum models.

We shall work in euclidean AdS_{d+1} and use Poincaré coordinates, which in the euclidean case cover the whole space. The metric reads

$$ds^2 = \frac{L^2}{z_0^2}\left(d\vec{z}^2 + dz_0^2\right), \tag{2.1}$$

where $\vec{z}$ is a vector with components z_i, $i = 1, \ldots, d$, $z_0 \geq 0$ and L is the AdS curvature. The dual field theory can be thought of as living on the boundary of this space, which is a d-dimensional sphere located at $z_0 = 0$ (together with the point at infinity). In the usual AdS/CFT correspondence, the holographic theory does not contain dynamical gravity. Essentially, the reason is that normalizable graviton modes in the bulk cannot reach the boundary of AdS, due to the divergent behaviour of the metric as $z_0 \to 0$. Moreover, the metric does not induce a determined metric at the boundary, but a conformal class. From the point of view of the CFT, the graviton is an external source coupling linearly to the stress-energy tensor. It does not propagate.

In RS2 a d-dimensional brane is located at $z_0 = \epsilon$ for some $\epsilon \geq 0$ and space-time is restricted to $z_0 \geq \epsilon$. Actually, in the standard RS2 one glues another copy of this semi-infinite AdS space and imposes a Z_2 symmetry, with $z_0 = \epsilon$ a fixed point of the orbifold, but for simplicity we shall work with just one copy of the amputated AdS, with orbifold boundary conditions at $z_0 = \epsilon$. Our results then differ by a factor 1/2 from the ones in the complete orbifold (for fields that are even under the Z_2). Note that the actual value of ϵ is not relevant, since we can always redefine the coordinates $\vec{z}$ to set $\epsilon = L$. However, we shall be interested in an expansion in ϵ, so it is convenient to keep it independent of L. An important feature of this construction is that a well-defined metric is induced on this brane (usually called Planck brane) and that normalizable graviton modes can reach it. The same holds for other bulk fields. Therefore, the holographic dual of RS2 is expected to contain d-dimensional dynamical gravity and, in general, propagating degrees of freedom corresponding to the different bulk fields. On the other hand, the Planck brane abruptly terminates the space beyond $z_0 = \epsilon$. The ultraviolet/infrared (UV/IR) correspondence [41] indicates that this corresponds to some kind of cutoff in the holographic theory. Let us make these ideas more explicit.

The dynamical content of the AdS/CFT correspondence is given by the identity [6, 7]

$$W^{\mathrm{CFT}}[\varphi] = S^{\mathrm{eff}}[\varphi], \tag{2.2}$$

where

$$W^{\rm CFT}[\varphi] = -\ln \int \mathcal{D}A \, \exp\left\{-S^{\rm CFT}[A] + \varphi\mathcal{O}\right\},$$
$$S^{\rm eff}[\varphi] = -\ln \int_{\phi(0)=\varphi} \mathcal{D}\phi \, \exp\{-S[\phi]\}\,. \tag{2.3}$$

Here, ϕ represents any of the fields propagating in AdS (in particular, the graviton), $\mathcal{O}$ is the corresponding dual operator in the CFT, φ is a field on the boundary manifold which acts as a source for $\mathcal{O}$, and A stands for all the fields in the CFT. $S^{\rm CFT}$ and $W^{\rm CFT}$ are, respectively, the action and the generating functional of connected correlation functions in the CFT, whereas S is the action of the gravity theory. The corresponding effective action, $S^{\rm eff}$, is a functional of φ defined by a path integral — we use field theory notation — in which the fields are constrained to take definite values at the boundary. We are interested in situations in which classical gravity is a good description of RS2, which corresponds to strong 't Hooft coupling and large N on the CFT side (if the CFT is a gauge theory, as in the known examples of the correspondence). In this case, $S^{\rm eff}$ reduces to the classical on-shell action, subject to the constraint that the boundary values of the fields ϕ coincide with φ.

As a matter of fact, both sides of the relation (2.2) are ill-defined. The on-shell gravity action is divergent in the IR whereas the correlation functions obtained from $W^{\rm CFT}$ contain UV divergencies at coincident points (again, we see the UV/IR correspondence at work). In order to make sense of this identity we introduce an IR regulator in the AdS theory and a UV regulator in the CFT. In particular, we can introduce an IR cutoff in AdS restricting the space to the range $z_0 \geq \epsilon$, just as in RS2. This corresponds to some definite (unknown) UV regularization on the CFT side. Then, the regularized AdS/CFT correspondence states that

$$W_\epsilon^{\rm CFT}[\varphi] = S_\epsilon^{\rm eff}[\varphi] \tag{2.4}$$

with

$$W_\epsilon^{\rm CFT}[\varphi] = -\ln \int_\epsilon \mathcal{D}A \, \exp\left\{-S^{\rm CFT}[A] + \varphi\mathcal{O}\right\},$$
$$S_\epsilon^{\rm eff}[\varphi] = -\ln \int_{\phi(\epsilon)=\varphi} \mathcal{D}\phi \, \exp\{-S_\epsilon[\phi]\}\,, \tag{2.5}$$

where $S_\epsilon[\phi]$ is the action of the gravity theory with the space-time integral restricted to $z_0 \geq \epsilon$, and the ϵ in the path integral indicates regularization. In practice we shall regularize the correlation functions obtained from functional differentiation of the unregularized generating functional. The regularized generating functional is then defined by its expansion in regularized correlation functions. This modification of the correspondence is relevant for Randall-Sundrum models because the effective theory of RS2 on the Planck brane is nothing but (two copies of) the r.h.s. of (2.4). Therefore, RS2 is holographically described by a *regularized* CFT.

Since we have just given heuristic arguments to obtain (2.4) from (2.2), one may have some doubts about the validity of this proposal. Furthermore, the effective theory of RS2 is known to contain dynamical gravity and it is not clear how kinetic terms for ϕ appear in $W_\epsilon^{\rm CFT}$. In order to solve this apparent problem and to better justify the identity (2.4), we follow [9, 10, 11] and consider the so-called renormalized AdS/CFT correspondence [37]–[40]. As a by-product we will find the alternative (standard) interpretation of the holographic theory as a renormalized CFT coupled to dynamical gravity. The renormalized AdS/CFT correspondence reads

$$W_{\rm ren}^{\rm CFT}[\varphi] = S_{\rm ren}^{\rm eff}[\varphi]\,. \tag{2.6}$$

The r.h.s. is defined by

$$S_{\rm ren}^{\rm eff}[\varphi] = \lim_{\epsilon\to 0}\left(S_\epsilon^{\rm eff}[\varphi] + S_\epsilon^{\rm ct}[\varphi]\right), \tag{2.7}$$

where $S_\epsilon^{\rm ct}$ contains the counterterms required for a well-defined $\epsilon \to 0$ limit. It has been shown in [37, 38, 39] that these counterterms can be written as functionals of the boundary fields. On the other hand, a renormalized CFT is generically defined by the addition of local UV counterterms:

$$\begin{aligned} W_{\rm ren}^{\rm CFT}[\varphi] &= -\lim_{\epsilon'\to 0}\ln\int_{\epsilon'}\mathcal{D}A\,\exp\left\{-S^{\rm CFT}[A] + \varphi\mathcal{O} - S_{\epsilon'}^{\rm CFT\,ct}[A,\varphi]\right\} \\ &= \lim_{\epsilon'\to 0}\left(W_{\epsilon'}^{\rm CFT}[\varphi] + W_{\epsilon'}^{\rm CFT\,ct}[\varphi]\right), \end{aligned} \tag{2.8}$$

where the prime in ϵ' indicates that this regularization may be unrelated to the one in the AdS theory. $W_{\epsilon'}^{\rm CFT\,ct}$ can be non local. Equations (2.7) and (2.8) define valid renormalization procedures, and it has been checked that (2.6) holds in some examples (in particular, the same Weyl anomaly is obtained in both theories [37, 42]). Note that both sides of the relation are scheme dependent, since the finite part of the counterterms is not fixed. Now, we can go one step back, remove the limits and state that

$$\begin{aligned} S_\epsilon^{\rm eff}[\varphi] &= W_{\epsilon'}^{\rm CFT}[\varphi] + W_{\epsilon'}^{\rm CFT\,ct}[\varphi] - S_\epsilon^{\rm ct}[\varphi] \\ &\equiv W_\epsilon^{\rm CFT}[\varphi]\,. \end{aligned} \tag{2.9}$$

The first identity is a consequence of (2.6), up to terms that vanish when $\epsilon \to 0$ and in the second one we have simply defined a new regularization of the CFT by adding specific counterterms to the generating functional. We see that a regularization scheme for the CFT exists such that (2.4) is fulfilled, at least to order ϵ^0. To this order, (2.9) can also be written as

$$S_\epsilon^{\rm eff}[\varphi] = W_{\rm ren}^{\rm CFT}[\varphi] - S_\epsilon^{\rm ct}[\varphi]\,. \tag{2.10}$$

Although $S_\epsilon^{\rm ct}$ can be non local, we shall see that the r.h.s. of (2.10) can be written as a path integral weighted by the exponential of minus a local action $S^{\rm CFT}[A]+S'[A,\varphi]$. This gives an alternative holographic interpretation of RS2 as a *renormalized* CFT coupled to the theory S'. Note that $S_\epsilon^{\rm ct}$, and then S'_ϵ, contain in particular the d-dimensional Einstein-Hilbert action. Of course, the very same $-S_\epsilon^{\rm ct}$ arises in the ϵ regularization of the CFT. Hence, an appropriately regularized CFT automatically contains dynamical d-dimensional gravity and, more generally, dynamical d-dimensional fields corresponding to all bulk fields that do not vanish on the brane. In the next sections we shall check explicitly that this is indeed the case. One should keep in mind that (2.10) is only a good approximation for small enough ϵ, which as we shall see corresponds to small momenta compared with the inverse AdS curvature. However, there is a double counting in thinking of RS2 as a cutoff CFT coupled to dynamical gravity.

Since all these arguments are independent of any modification of the theory at large values of z_0, the same conclusions apply to RS1. Of course, in this case the second brane breaks conformal invariance in the IR.

3. Correlations functions in RS2

In this section we calculate correlation functions in RS2 for the fields induced on the Planck brane, using perturbation theory in the coupling constants. For simplicity we consider only scalar fields but analogous results hold for fields with higher spin. Moreover, we assume that these scalars move in a fixed background, i.e. we ignore the back-reaction on the metric of the scalars. In our perturbative formalism, this would be described as higher-point correlators of scalars and gravitons in linearized gravity.

3.1 General formalism

To simplify the notation, we start with a single scalar field, ϕ. Let $\varphi(\vec{z}) = \phi(\epsilon,\vec{z})$. We are interested in the effective theory for the boundary field φ. The correlation functions of several fields ϕ can be obtained from the generating functional

$$\begin{aligned} Z[j] &= \int \mathcal{D}\phi \, \exp\left\{-S[\phi] + \int_{\rm brane} \varphi j\right\} \\ &= \int \mathcal{D}\varphi \exp\left\{-S^{\rm eff}[\varphi] + \int_{\rm brane} \varphi j\right\}, \end{aligned} \tag{3.1}$$

where $S^{\rm eff}[\varphi]$ is defined as in (2.5). From now on the subindex ϵ is implicit. We have splitted the path integral into two parts: first, we calculate the effective action by integrating out the bulk degrees of freedom with Dirichlet boundary conditions; second, we use it to calculate $Z[j]$. $S^{\rm eff}$ contains non-local interactions. For classical induced fields, it coincides with the generating functional of 1PI correlation functions.

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Therefore,

$$\langle \varphi(x_1) \cdots \varphi(x_n) \rangle^{1\mathrm{PI}} = \left[\frac{\delta}{\delta\varphi(x_1)} \cdots \frac{\delta}{\delta\varphi(x_n)} S^{\mathrm{eff}}[\varphi] \right]_{\varphi=0} , \tag{3.2}$$

and the relation (2.4) implies

$$\langle \varphi(x_1) \cdots \varphi(x_n) \rangle^{1\mathrm{PI}} = \langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle . \tag{3.3}$$

The r.h.s. is a connected correlation function in the dual CFT. The connected correlations functions of boundary fields are obtained in the standard way from the 1P1 ones. Observe that (3.3) means that, as expected for dual fields, the generating functionals of φ and $\mathcal{O}$ correlation functions are related by a Legendre transformation.

At this point, one might worry about the relation between the correlators calculated from S^{eff}, which are obtained with Dirichlet boundary conditions, and the propagators calculated in [10, 43], which used Neunman boundary conditions. The analysis below is an extension of the discussion in [10]. In Randall-Sundrum models one must impose Neumann boundary conditions at the orbifold fixed points for fields that are even under the Z_2 symmetry. This is implied by the orbifold structure and continuity of the derivative of the fields at the branes.[1]

As it was argued in [10], both approaches arise from two different ways of splitting the path integral. In (3.1) the bulk fields are integrated with Dirichlet boundary conditions, so that one must still perform the integration over the brane fields. We shall refer to this procedure as the "D approach". One could also integrate first the fields in a neighbourhood of the brane, which imposes the constraint that fields should obey Neumann boundary conditions at the brane. The remaining path integral over the bulk fields must be carried out with this constraint. We call this the "N approach". Let us consider the classical limit. In the N approach, the on-shell field generated by a source j located on the brane can be written as

$$\phi(z_0, \vec{z}) = \int d^d x \sqrt{g(\epsilon)} \Delta_N(\epsilon, \vec{x}; z_0, \vec{z}) j(\vec{x}) , \tag{3.4}$$

where $g(\epsilon)$ is the determinant of the $(d+1)$-dimensional metric evaluated on the brane and Δ_N is the scalar Neumann propagator of [10, 43]. This is the Green function of the quadratic equation of motion with a (bulk) source $J(y)$, with boundary conditions $[\partial_0 \Delta_N(y_0, \vec{y}; z_0, \vec{z})]_{z_0=\epsilon} = 0$, $y_0 \neq \epsilon$, and $\lim_{z_0\to\infty} \Delta_N(y_0, \vec{y}; z_0, \vec{z}) = 0$. We use the notation $\partial_0 = \partial/\partial z_0$. The propagator $\Delta_N(\epsilon, \vec{x}; z_0, \vec{z})$ is defined from the limit $y_0 \to \epsilon$; we note that it does not obey the Neumann condition on the brane. We use in (3.4) the full metric, instead of the induced metric, in order to have the same normalization for the propagator as in [10]. Sources inside the bulk will be considered in section 6.

[1] In some important cases the action contains terms proportional to delta functions at the branes that make the field derivatives discontinuous, and one must add appropriate corrections. We comment on this below. On the other hand, odd fields must vanish at the Planck brane and do not appear in the effective description.

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In the D approach, the on-shell field is written as

$$\phi(z_0,\vec{z}) = \int d^dx \mathcal{K}(\vec{x};z_0,\vec{z})\varphi(\vec{x})\,, \tag{3.5}$$

where $\mathcal{K}$ is the Dirichlet "bulk-to-boundary" propagator, i.e. a solution to the homogeneous equation of motion obeying $\mathcal{K}(\vec{x};\epsilon,\vec{z}) = \delta^{(d)}(\vec{x}-\vec{z})$ and $\lim_{z_0\to\infty}\mathcal{K}=0$. Observe that the field $\varphi(\vec{x})$ is still off shell. As is customary in AdS/CFT we have defined $\mathcal{K}$ such that the unit metric appears in the integral (3.5). Both approaches must be equivalent after complete path integration. Therefore, both expressions for ϕ must agree for on-shell φ, that is to say, when the φ equation of motion, derived from S^{eff}, is imposed. Indeed, in the next subsection we show explicitly that the propagator of the theory S^{eff} is precisely $\Delta_N(\epsilon,\vec{x};\epsilon,\vec{y})$. Hence, on shell,

$$\varphi(\vec{x}) = \int d^dy\sqrt{g(\epsilon)}\Delta_N(\epsilon,\vec{x};\epsilon,\vec{y})j(y) \tag{3.6}$$

and

$$\begin{aligned}\phi(z_0,\vec{z}) &= \int\int d^dx d^dy\,\sqrt{g(\epsilon)}\mathcal{K}(\vec{x};z_0,\vec{z})\Delta_N(\epsilon,\vec{x};\epsilon,\vec{y})j(y)\\ &= \int d^dy\sqrt{g(\epsilon)}\Delta_N(\epsilon,\vec{y};z_0,\vec{z})j(\vec{y})\,,\end{aligned} \tag{3.7}$$

which agrees with (3.4). We have used the identity

$$\int d^dx\Delta_N(\epsilon,\vec{x};\epsilon,\vec{y})\mathcal{K}(\vec{x};z_0,\vec{z}) = \Delta_N(\epsilon,\vec{y};z_0,\vec{z})\,, \tag{3.8}$$

which can be checked by explicit computation. So we see that both procedures give the same bulk fields when the brane fields satisfy their classical equations of motion. As we mentioned in the last footnote, in some cases the action contains quadratic terms proportional to delta functions (the so-called boundary mass terms) [44]. In the N approach one has to modify the Neunman boundary condition to take into account the discontinuity of the field derivatives. In the D approach, the effective action will contain these boundary mass terms, which change the on-shell φ in such a way that the same bulk field is obtained again. In the following we shall be interested in the value of $S^{\mathrm{eff}}[\varphi]$ for off-shell φ, from which the correlation functions of these fields can be obtained. Since correlation functions of φ are nothing but correlation functions of ϕ for points on the brane, a direct consequence of this analysis is that correlation functions calculated in the N approach are related to (regularized) Witten diagrams by a Legendre transformation. As in usual field theory, 1PI correlators are easier to calculate and contain all the information of the theory. Here we are simply comparing two methods for calculating the same object in the AdS theory, with no reference to any CFT. But of course, we are interested in the D procedure because it allows a direct comparison with the holographic dual.

3.2 Two-point functions

Consider a scalar field ϕ of mass M^2 in AdS_{d+1}. The quadratic part of the euclidean action reads

$$S = \frac{1}{2}\int_\epsilon d^{d+1}z\sqrt{g}\left(\partial^\mu\phi\partial_\mu\phi + M^2\phi^2\right). \tag{3.9}$$

where $\int_\epsilon d^{d+1}z = \int d^dz \int_\epsilon^\infty dz_0$. The normalization of ϕ has been fixed to obtain a canonical kinetic term. The equation of motion reads

$$z_0^{d+1}\partial_0\left(z_0^{-d+1}\partial_0\phi(z_0,\vec{k})\right) - (k^2 z_0^2 + m^2)\phi(z_0,\vec{k}) = 0\,, \tag{3.10}$$

where we have Fourier transformed the field in the coordinates tangent to the brane:

$$\phi(z_0,\vec{k}) = \int d^dz\, e^{i\vec{k}\cdot\vec{z}}\phi(z_0,\vec{z})\,. \tag{3.11}$$

Unless otherwise indicated, scalar products are defined with the flat euclidean metric, which is related to the induced metric by a constant Weyl transformation: $g_{\text{ind}\,ij} = L^2/\epsilon^2\,\delta_{ij}$. Translation invariance along the brane implies that the momentum $\vec{k}$ is conserved. We have also defined the dimensionless mass $m^2 = L^2M^2$. This squared mass can be negative, but must obey the Breitenlohner-Freedman bound $m^2 \geq -d^2/4$. The conformal dimension of the field (which determines its boundary behaviour) is $\Delta = 1/2(d+\sqrt{d^2+4m^2}) \equiv \nu + d/2$.[2] The particular solutions of this differential equation are $z_0^{d/2}I_\nu(z_0k)$ and $z_0^{d/2}K_\nu(z_0k)$, where I_ν and K_ν are modified Bessel functions and $k=|\vec{k}|$. Regularity in the interior selects $\phi \propto z_0^{d/2}K_\nu(z_0k)$. The Dirichlet bulk-to-boundary propagator is a solution such that

$$\lim_{z_0\to\epsilon}\mathcal{K}(z_0,\vec{k}) = 1\,; \qquad \lim_{z_0\to\infty}\mathcal{K}(z_0,\vec{k}) = 0\,. \tag{3.12}$$

The explicit form of this propagator is

$$\mathcal{K}(z_0,\vec{k}) = \left(\frac{z_0}{\epsilon}\right)^{d/2}\frac{K_\nu(kz_0)}{K_\nu(k\epsilon)}\,. \tag{3.13}$$

The on-shell bulk field is then $\Phi(z_0,\vec{k}) = \mathcal{K}(z_0,\vec{k})\varphi(\vec{k})$. Inserting this expression in the action, integrating by parts and using the equation of motion, the (on-shell) action reduces to a surface term (see [45], for instance). Double functional differentiation with respect to φ yields

$$\langle\varphi(\vec{k})\varphi(\vec{k}')\rangle^{1\text{PI}} = \delta(\vec{k}+\vec{k}')L^{-1}\left(\frac{L}{\epsilon}\right)^d\left(\nu - \frac{d}{2} + \frac{k\epsilon K_{\nu-1}(k\epsilon)}{K_\nu(k\epsilon)}\right). \tag{3.14}$$

[2] For $-d^2/4 < m^2 \leq -d^2/4+1$ there are two alternative AdS-invariant quantizations, one with Δ as defined above and the other with $\Delta_- = 1/2(d-\sqrt{d^2+4m^2})$. In this paper we always consider the first possibility. We shall make some comments about the second one below.

From now on we absorb the momentum conservation deltas into the definition of the correlation functions. The connected two-point correlator (the propagator of the effective theory) is simply the inverse of the 1PI one:

$$\langle\varphi(\vec{k})\varphi(-\vec{k})\rangle = L\left(\frac{\epsilon}{L}\right)^d \frac{K_\nu(k\epsilon)}{(\nu-d/2)K_\nu(k\epsilon)+k\epsilon K_{\nu-1}(k\epsilon)}\,. \tag{3.15}$$

It is instructive to study this propagator in Minkowski space. We use $(-1,1,\ldots,1)$ signature. Taking provisionally $\epsilon = L$ and performing a Wick rotation, the propagator (3.15) reads

$$\langle\varphi(\vec{k})\varphi(-\vec{k})\rangle = -L\,\frac{H^{(1)}_\nu(qL)}{(\nu-d/2)H^{(1)}_\nu(qL)-qLH^{(1)}_{\nu-1}(qL)}\,, \tag{3.16}$$

where $H^{(1)} = J+iY$ is the first Hankel function and $q^2 = -k^\mu k_\mu$. For timelike k, $q^2>0$. This expression agrees with the scalar Neumann propagators calculated in [10, 43] (for both points on the brane). Actually, in Minkowski space one can add a normalizable contribution that modifies this result [46]. The Minkowskian propagator that results from Wick rotating the euclidean one is the one satisfying Hartle-Hawking boundary conditions near the horizon. The relation of this propagator with the Kaluza-Klein description of [4] is given by the spectral representation

$$\langle\varphi(\vec{k})\varphi(-\vec{k})\rangle = \int_0^\infty d\mu \frac{\sigma(\mu)}{k^2-\mu+i\varepsilon}\,. \tag{3.17}$$

Here, $\sigma(\mu)$ is the amplitude at the Planck brane of the wave function of a Kaluza-Klein mode with mass squared μ. This equation can be inverted to give

$$\sigma(\mu) = -\pi\,\mathrm{Im}\left[\langle\varphi(\vec{k})\varphi(-\vec{k})\rangle\right]_{q^2=\mu+i\varepsilon}\,. \tag{3.18}$$

The $\sigma(\mu)$ thus calculated is positive definite. We have checked numerically that the propagator has no poles for timelike or null $\vec{k}$ except in the massless case. A zero mass corresponds to $\nu = d/2$ and one can use a Bessel recursion relation to show that in this case [10]

$$\langle\varphi(\vec{k})\varphi(-\vec{k})\rangle = -L\left(\frac{d-2}{q^2}-\frac{1}{q}\frac{H^{(1)}_{d/2-2}(qL)}{H^{(1)}_{d/2-1}(qL)}\right). \tag{3.19}$$

We have isolated the part with the pole, which is the standard propagator of a massless scalar field. This corresponds to the zero mode in the Kaluza-Klein decomposition. The rest, which is suppressed at low energies by powers of q/L, contains no poles and corresponds to a continuum (with no mass gap) of Kaluza-Klein modes. Scalars with a positive mass, on the other hand, contain no isolated mode in their d-dimensional description. Nevertheless, we shall see that the leading term of their propagator in a low energy expansion is a standard massive d-dimensional propagator. Finally, there is an isolated pole for spacelike $\vec{k}$. Hence, the spectrum contains a tachyon. This instability has been found in [47] using both the $d+1$ propagator and a Kaluza-Klein approach. The authors of this reference then argue that the instability might be cured by the CFT in a holographic interpretation. This does not make sense since the CFT must give the same effects as the AdS theory. They also comment that the instability may disappear when the back-reaction of the scalar on the metric is taken into account. The results in [2, 48] show that this is indeed the case.

In order to compare with the holographic theory we expand the correlation functions about $\epsilon = 0$. In general we have to distinguish between fields with integer and with non-integer index ν, as the expansion of K_ν contains logarithms for integer ν. Let us start with the non-integer case. The 1PI two-point function (3.14) has the expansion

$$\langle\varphi(\vec{k})\varphi(\vec{k'})\rangle^{\mathrm{1PI}} = L^{-1}\left(\frac{L}{\epsilon}\right)^d \left[\left(\nu - \frac{d}{2}\right) + \frac{k^2\epsilon^2}{2(\nu-1)} \frac{1+\sum_{n=1}^{[\nu]-1} \frac{(-1)^n\Gamma(\nu-1-n)}{4^n n!\Gamma(\nu-1)}(k\epsilon)^{2n}}{1+\sum_{n=1}^{[\nu]-1} \frac{(-1)^n\Gamma(\nu-n)}{4^n n!\Gamma(\nu)}(k\epsilon)^{2n}} + \right.$$
$$\left. + \frac{2\Gamma(1-\nu)}{4^\nu\Gamma(\nu)}(k\epsilon)^{2\nu}\right] + O\left(\epsilon^{2[\nu]-d+2}\right), \tag{3.20}$$

where $[\nu]$ denotes the entire part of ν and the sums are understood to vanish whenever the upper index is smaller than the lower one. Note that the expansion in ϵ is equivalent to a derivative expansion. The leading non-local part is proportional to $k^{2\nu}$. After Wick rotation, it gives the leading contribution to the imaginary part of the Minkowskian two-point function, with the correct sign. This non-local part is the one calculated in standard AdS/CFT calculations. Since for $\epsilon = 0$ (and $\nu \neq d/2$) the fields ϕ diverge or vanish at the real boundary of AdS, finite correlation functions are obtained by rescaling the operators in such a way that they do not couple to $\phi(0)$, but rather to $\lim_{z_0\to 0} z_0^{\nu-d/2}\phi(z_0)$, which is finite. In the regularized AdS/CFT correspondence it is convenient to normalize the operators in the same way:

$$\langle\mathcal{O}_{\Delta_1}\cdots\mathcal{O}_{\Delta_n}\rangle = \epsilon^{nd-(\Delta_1+\cdots+\Delta_n)}\langle\varphi_{\Delta_1}\cdots\varphi_{\Delta_n}\rangle^{\mathrm{1PI}}. \tag{3.21}$$

(For simplicity we ignored this subtlety in section 3.1.) Then the leading non-local term of (3.20) gives an ϵ-independent contribution to correlation functions of operators. In the standard AdS/CFT ($\epsilon \to 0$) this term yields a conformal finite non-local expression, while the local terms are divergent and must be cancelled by appropriate counterterms [37, 38, 39].

The expansion (3.20) also shows that the leading contribution to the φ propagator is just the usual propagator of a d-dimensional scalar field with squared mass $(m^{\mathrm{eff}})^2 = 2(\nu - d/2)(\nu-1)/\epsilon^2$. For $\epsilon = L$ this is of the same order as the mass of the bulk field. Both masses are expected to be of the Planck mass order. The higher-derivative corrections smooth the behaviour at $k^2 \sim -(m^{\rm eff})^2$ and remove the pole (except in the case $\nu < d/2$, in which the pole remains). So, for $\nu > d/2$, φ does not describe a four-dimensional particle, but rather a continuous spectral density roughly peaked at $(m^{\rm eff})^2$. It is important to note that boundary mass terms can change this and give rise to a pole in the propagator. In the Kaluza-Klein description, the reason is that the additional delta functions at the brane can, in some cases, support a bounded state in the equivalent quantum-mechanical problem. For $\nu < d/2$ there is an isolated space-like pole plus a time-like continuum.

From (3.20) we see that the quadratic part of the effective action has the form

$$\begin{aligned} S^{\rm eff} &= \frac{1}{2L}\left(\frac{L}{\epsilon}\right)^d \int d^dx\, d^dy\, \varphi(\vec{x}) \Big[a_0\delta(\vec{x}-\vec{y}) + a_1\epsilon^2\Box\delta(\vec{x}-\vec{y}) + \cdots + \\ &\qquad + a_{[\nu]}\epsilon^{2[\nu]}\Box^{[\nu]}\delta(\vec{x}-\vec{y}) + \\ &\qquad + b\epsilon^{2\nu}\Box^{[\nu]+1}\frac{1}{|\vec{x}-\vec{y}|^{2(\nu-[\nu]+d/2-1)}} + \cdots\Big]\varphi(\vec{y}) \\ &= \frac{1}{2L}\int d^dx\sqrt{g_{\rm ind}}\varphi(\vec{x})\ \Big[a_0\delta(\vec{x}-\vec{y}) + a_1L^2\Box_{\rm ind}\delta(\vec{x}-\vec{y}) + \cdots + \\ &\qquad + a_{[\nu]}L^{2[\nu]}\Box^{[\nu]}_{\rm ind}\delta(\vec{x}-\vec{y}) + \\ &\qquad + bL^{2\nu}\Box^{[\nu]+1}_{\rm ind}\frac{1}{((x^i-y^i)(x_i-y_i))^{(\nu-[\nu]+d/2-1)}} + \cdots\Big]\varphi(\vec{x})\,, \end{aligned} \tag{3.22}$$

where $x^ix_i = g^{\rm ind}_{ij}x^ix^j$ and $\Box_{\rm ind} = g^{ij}_{\rm ind}\partial_i\partial_j$. The non-local term is nothing but a renormalized expression of $|\vec{x}-\vec{y}|^{-2\Delta}$. The regularization and renormalization of these expressions is discussed in section 4 and some useful formulae are collected in appendix A. As emphasized in [39], the effective action does not depend explicitly on ϵ when expressed in terms of the induced metric and scalar field. This gives the physical meaning of the expansion in ϵ for an observer living on the brane: it corresponds to a low-energy expansion with energies measured in units of L^{-1}. Note that we can rescale the brane coordinates such that $g_{\rm ind} = \delta$ (equivalently, $\epsilon = L$).

Consider now an integer ν. For $\nu \geq 1$ the expansion of the two-point function is the same as above, but with the $\epsilon^{2(\nu-d/2)}$ term given by

$$\frac{(-1)^\nu 4^{1-\nu}}{\Gamma(\nu)^2}k^{2\nu}\epsilon^{2(\nu-d/2)}\ln(k\epsilon) + c\,k^{2\nu}\epsilon^{2(\nu-d/2)}\,, \tag{3.23}$$

with c a constant number. In the renormalized AdS/CFT correspondence, this logarithm has to be cancelled by a counterterm proportional to $\log M\epsilon$, with M an arbitrary scale. This produces an anomalous breaking of conformal invariance in the CFT which agrees with the violation of conformal invariance usually introduced in the renormalization procedure [37, 42, 74]. The effective mass has the same expression as in the non-integer case. It vanishes if and only if the bulk field is massless ($\nu = d/2$). In this case, the pole at $k^2 = 0$ is obviously preserved by the corrections and φ contains a discrete mode.

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The case $\nu = 0$ corresponds to the smallest mass allowed by unitarity, $m^2 = -d^2/4$, and has some special features. The two-point function reads in this case

$$\langle \varphi(\vec{k})\varphi(-\vec{k})\rangle^{1\mathrm{PI}} = L^{-1}\left(\frac{L}{\epsilon}\right)^d \left[-\frac{d}{2} - \frac{1}{\ln(k\epsilon\gamma_E/2)}\right], \tag{3.24}$$

where $\gamma_E = 1.781\ldots$ is the Euler constant. The appearance of a logarithm in the denominator is related to the fact that the bulk field diverges as $z_0^{d/2}\ln(z_0 k)$ near the AdS boundary. It seems reasonable to define the correlation function of two operators of conformal dimension $d/2$ by [49]

$$\langle \mathcal{O}(\vec{k})\mathcal{O}(-\vec{k})\rangle = \left(\epsilon^{d/2}\ln(\epsilon M')\right)^2 \langle \varphi(\vec{k})\varphi(-\vec{k})\rangle^{1\mathrm{PI}} \tag{3.25}$$

for some scale M'. $\langle \mathcal{O}\mathcal{O}\rangle$ then contains the correct finite non-local term of the CFT, $\ln(kM')$, with corrections of order $1/\ln(\epsilon M')$ that vanish logarithmically as $\epsilon \to 0$. In the local terms, on the other hand, (divergent) double logarithms appear. This is not expected from the CFT analysis in section 4. Furthermore, we have found that the extension of (3.25) to three-point functions does not even give the correct non-local part. Therefore, the regularized AdS/CFT needs some modification in this particular case. This may be related to the fact that the Dirichlet propagator defined in (3.13) cannot describe fields with dimension $(d-2)/2 \le \Delta < d/2$. Indeed, for a given mass, this propagator automatically selects the largest conformal dimension (Δ and not Δ_-). $\nu = 0$ is the special case where $\Delta = \Delta_-$. We will not attempt to find a correct prescription for $\Delta = d/2$ or $\Delta < d/2$ in this paper, but leave it as an interesting open problem.[3]

3.3 Three-point functions

Suppose now that the action contains a cubic term of the form

$$S \supset \frac{\lambda_{ijk}}{3!}\int_\epsilon d^{d+1}z\,\sqrt{g}\,\phi_i\phi_j\phi_k\,, \tag{3.26}$$

where ϕ_i is a scalar field of dimension Δ_i and the coupling λ_{ijk} is completely symmetric in the indices. In momentum space, the 1PI three-point functions in the effective theory read

$$\begin{aligned}\langle \varphi_1(\vec{k}_1)\varphi_2(\vec{k}_2)\varphi_3(\vec{k}_3)\rangle^{1\mathrm{PI}} = -\lambda_{123}\int_\epsilon^\infty dz_0\left(\frac{L}{z_0}\right)^{d+1}\mathcal{K}_{\nu_1}(z_0,\vec{k}_1)\mathcal{K}_{\nu_2}(z_0,\vec{k}_2)\,\times \\ \times\,\mathcal{K}_{\nu_3}(z_0,\vec{k}_3)\,. \end{aligned}\tag{3.27}$$

The integral over z_0 is difficult to perform in general. Fortunately, it is possible to compute in a simple manner the leading terms in the ϵ expansion. We distinguish

[3] This is only a problem of the cutoff regularization (which is the relevant one for Randall-Sundrum). The renormalization procedure proposed in [50] gives correct CFT correlations functions for $\Delta \le d/2$.

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local terms, for which all points are coincident (two delta functions in coordinate space), semilocal terms, for which two points coincide and the other is kept appart (one delta function), and completely non-local terms, for which all points are non coincident. We shall calculate all the terms up to the leading completely non-local term. First, observe that the integral has the form

$$\langle\varphi_1(\vec{k}_1)\varphi_2(\vec{k}_2)\varphi_3(\vec{k}_3)\rangle^{\rm 1PI} = -\lambda_{123}L^{d+1}\frac{G(\epsilon)}{f(\epsilon)}\,, \tag{3.28}$$

with

$$G(\epsilon) = \int_\epsilon^\infty dz_0\, z_0^{-d-1} f(z_0)\,. \tag{3.29}$$

Consider non-integer indices ν_1, ν_2 and ν_3. The function $f(z_0)$ can be expanded about $z_0 = 0$:

$$\begin{aligned} f(z_0) &= \prod_{i=1}^{3} \mathcal{K}^0_{\nu_i}(z_0,\vec{k}_i) \\ &= C_\nu \prod_{i=1}^{3} z_0^{d/2-\nu_i}\left(\sum_{n=0}^{\infty} a_n^{(i)} z_0^{2n} + \beta^{(i)} z_0^{2\nu_i}\sum_{n=0}^{\infty} b_n^{(i)} z_0^{2n}\right), \end{aligned} \tag{3.30}$$

where

$$\mathcal{K}^0_\nu(z_0,\vec{k}) = C_\nu k^\nu z_0^{d/2} K_\nu(z_0 k) \tag{3.31}$$

is the stantard $\epsilon = 0$ bulk-to-boundary propagator, $C_\nu = \frac{\sin\pi\nu}{2^{\nu-1}\pi}\Gamma(1-\nu)$, and the explicit expressions of the momentum-dependent coefficients are

$$\begin{aligned} a_n^{(i)} &= \frac{\Gamma(1-\nu_i)}{n!\Gamma(1-\nu_i+n)}\left(\frac{k_i}{2}\right)^{2n}, \\ b_n^{(i)} &= \frac{\Gamma(1+\nu_i)}{n!\Gamma(1+\nu_i+n)}\left(\frac{k_i}{2}\right)^{2n}, \\ \beta^{(i)} &= -\frac{\Gamma(1-\nu_i)}{n!\Gamma(1+\nu_i)}\left(\frac{k_i}{2}\right)^{2\nu_i}. \end{aligned} \tag{3.32}$$

Note that $a_0^{(i)} = b_0^{(i)} = 1$. Using (3.30) in (3.29),

$$\begin{aligned} G(\epsilon) = C_\nu \int_\epsilon^\infty dz_0 z_0^{d/2-1-\sigma} &\left[\sum_{n=0}^{\left[\frac{1}{2}(\sigma-d/2)\right]} \tilde{a}_n z_0^{2n} + \sum_{i=1}^{3}\left(\beta^{(i)} z_0^{2\nu_i} \sum_{n=0}^{\left[\frac{1}{2}(\sigma-d/2)-\nu_i\right]} \tilde{a}_n^{(i)} z_0^{2n}\right) + \right. \\ &\left. + \sum_{i\neq j=1}^{3}\left(\beta^{(i)}\beta^{(j)} z_0^{2(\nu_i+\nu_j)} \sum_{n=0}^{\left[\frac{1}{2}(\sigma-d/2)-\nu_i-\nu_j\right]} \tilde{a}_n^{(i,j)} z_0^{2n}\right)\right] + \\ &+ \tilde{G}(\epsilon)\,, \end{aligned} \tag{3.33}$$

where $\sigma = \sum_{i=1}^{3} \nu_i$ and

$$\begin{aligned}
\tilde{a}_n &= \sum_{n_1=0}^{n} \sum_{n_2=0}^{n-n_1} \sum_{n_3=0}^{n-n_1-n_2} a_{n_1}^{(1)} a_{n_2}^{(2)} a_{n_3}^{(3)} \,, \\
\tilde{a}_n^{(1)} &= \sum_{n_1=0}^{n} \sum_{n_2=0}^{n-n_1} \sum_{n_3=0}^{n-n_1-n_2} b_{n_1}^{(1)} a_{n_2}^{(2)} a_{n_3}^{(3)} \,, \\
\tilde{a}_n^{(1,2)} &= \sum_{n_1=0}^{n} \sum_{n_2=0}^{n-n_1} \sum_{n_3=0}^{n-n_1-n_2} b_{n_1}^{(1)} b_{n_2}^{(2)} a_{n_3}^{(3)} \,.
\end{aligned} \tag{3.34}$$

Other $\tilde{a}_n^{(i)}$ and $\tilde{a}_n^{(i,j)}$ are analogously defined. The rest $\tilde{G}$ is at most logarithmically divergent when $\epsilon \to 0$. Let us assume momentarily that the conformal dimensions of the fields are such that $\frac{1}{2}(\sigma - d/2)$, $\frac{1}{2}(\sigma - d/2 - \nu_i)$, and $\frac{1}{2}(\sigma - d/2 - \nu_i - \nu_j)$ are neither positive integers nor zero. Then, each of the terms that we have written explicitly in (3.33) vanishes sufficiently rapidly at infinity, and we can perform the corresponding integrals. Furthermore, in this case $\tilde{G}(\epsilon)$ is finite as $\epsilon \to 0$. We find

$$G(\epsilon) = G^{\text{div}}(\epsilon) + \tilde{G}(0) + O(\epsilon^{\eta}) \,, \tag{3.35}$$

with $\eta > 0$ and

$$\begin{aligned}
G^{\text{div}}(\epsilon) = C_\nu \frac{\epsilon^{d/2-\sigma}}{d/2-\sigma} \Bigg[& \sum_{n=0}^{\left[\frac{1}{2}(\sigma-d/2)\right]} \frac{(d/2-\sigma)\tilde{a}_n}{d/2-\sigma+2n} \epsilon^{2n} + \\
& + \sum_{i=1}^{3} \left(\beta^{(i)} \epsilon^{2\nu_i} \sum_{n=0}^{\left[\frac{1}{2}(\sigma-d/2)-\nu_i\right]} \frac{(d/2-\sigma)\tilde{a}_n^{(i)}}{d/2-\sigma+\nu_i+2n} \epsilon^{2n} \right) + \\
& + \sum_{i\neq j=1}^{3} \Bigg(\beta^{(i)} \beta^{(j)} \epsilon^{2(\nu_i+\nu_j)} \times \\
& \times \sum_{n=0}^{\left[\frac{1}{2}(\sigma-d/2)-\nu_i-\nu_j\right]} \frac{(d/2-\sigma)\tilde{a}_n^{(i,j)}}{d/2-\sigma+2\nu_i+2\nu_j+2n} z_0^{2n} \Bigg) \Bigg] \,.
\end{aligned} \tag{3.36}$$

The 1PI three-point function is obtained by using (3.30), (3.35) and (3.36) in (3.28). It has the form

$$\begin{aligned}
& \epsilon^{-d} \left[\sum_n \alpha_n \epsilon^{2n} + \sum_i \beta^{(i)} \sum_n \alpha_n^{(i)} \epsilon^{2(\nu_i+n)} + \sum_{i,j} \beta^{(i)} \beta^{(j)} \sum_n \alpha_n^{(i,j)} \epsilon^{2(\nu_i+\nu_j+n)} \right] + \\
& + \epsilon^{\sigma-3\frac{d}{2}} \tilde{G}(0) + O\left(\epsilon^{\sigma-3\frac{d}{2}+\eta}\right) .
\end{aligned} \tag{3.37}$$

All the terms inside the bracket are either local or semilocal, except the ones with $\beta^{(i)}\beta^{(j)}$, $i \neq j$, which are proportional to $k_i^{2\nu_i} k_j^{2\nu_j}$. To order $\epsilon^{\sigma - 3\frac{d}{2}}$ these terms only contribute when $\sigma > d/2 - \nu_i - \nu_j$ (remember that we have assumed $\sigma \neq d/2 - \nu_i - \nu_j$ for the moment), i.e. when one of the conformal dimensions, Δ_i, is greater than the sum of the others. The coefficients λ_{ijk} of such "superextremal" functions vanish in type-IIB supergravity on AdS_5 compactified on S^5, due to the properties of the S^5 spherical harmonics. (In the dual $\mathcal{N} = 4$ theory these functions are forbidden by R symmetry.) More generally, for any theory on AdS_{d+1}, the completely non-local part of superextremal n-point functions diverges — after the rescaling (3.21) — when $\epsilon \to 0$. This invalidates the (unregulated) AdS/CFT correspondence unless $\lambda_{ijk} = 0$ when $\Delta_i > \Delta_j + \Delta_k$. For these reasons, we assume that λ_{ijk} does vanish in any superextremal case. On the other hand, the terms with one $\beta^{(i)}$ are semilocal. These terms are relevant in our approximation when $\sigma > d/2 + 2\nu_i$, and in this case are divergent after rescaling when $\epsilon \to 0$. One might be worried about the fact that one needs to add non-local counterterms to $S^{\rm eff}$ in order to obtain a finite renormalized effective action. However, this is not so strange since $S^{\rm eff}$ itself is non local. Presumably, the non-local counterterms arise from local counterterms in the original action when the bulk degrees of freedom are integrated out. The holographic counterpart of this renormalization procedure is discussed in section 4.

The leading completely non-local term is given by $\tilde{G}(0)$. It can be calculated in coordinate space using conformal techniques [45]. The result is

$$\frac{c}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{13}^{\Delta_1+\Delta_3-\Delta_2} x_{23}^{\Delta_2+\Delta_3-\Delta_1}} \tag{3.38}$$

with

$$\begin{aligned} c = & -\frac{\Gamma\left[\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3)\right]\Gamma\left[\frac{1}{2}(\Delta_1+\Delta_3-\Delta_2)\right]\Gamma\left[\frac{1}{2}(\Delta_2+\Delta_3-\Delta_2)\right]}{2\pi^d\Gamma\left[\Delta_1-d/2\right]\Gamma\left[\Delta_1-d/2\right]\Gamma\left[\Delta_1-d/2\right]} \times \\ & \times \Gamma\left[\frac{1}{2}(\Delta_1+\Delta_2+\Delta_3-d)\right]. \end{aligned} \tag{3.39}$$

We have defined $x_{ij} = |\vec{x}_i - \vec{x}_j|$. The functional form in (3.38) is dictated by conformal invariance. As it stands, this expression is valid only for non-coincident points. $\tilde{G}(0)$ is given by its renormalized value. In order to find the three-point functions of CFT operators we must rescale according to (3.21).

We have found the 1PI three-point function to order $\epsilon^{\sigma - 3\frac{d}{2}}$, at least in coordinate space and excluding some particular values of the conformal dimensions. These exceptions are interesting. They correspond to either the presence of logarithms in the expansion of $f(z_0)$ (when at least one ν_i is integer) or to $1/z_0$ terms in the integrand of (3.33), which also give rise to logarithms. In the first case, it is straightforward to modify (3.36) to incorporate the logarithms in the expansion of $f(z_0)$. Let us see how to deal with the second possibility. The integral of the $1/z_0$ terms diverges at infinity

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and cannot be performed independently. Instead, we can split the logarithmically divergent integrals in the following way:

$$\int_{\epsilon}^{\infty} dz_0\, z_0^{-1} = -\ln(M\epsilon) + \int_{M^{-1}}^{\infty} dz_0\, z_0^{-1}\,. \tag{3.40}$$

The second term can then be included in $\tilde{G}(\epsilon)$, which is a well-defined integral because the sum of all terms is well-behaved at infinity. Moreover, $\tilde{G}(\epsilon)$ is convergent for $\epsilon \to 0$ and can be written as $\tilde{G}(0) + O(\epsilon^{\eta})$. Since the complete $G(\epsilon)$ does not depend on the arbitrary scale M, $\tilde{G}(0)$ must contain terms of the form $\ln(k/M)$. Again, $\tilde{G}(0)$ can be obtained from an standard $\epsilon = 0$ AdS/CFT calculation. As discuss in the next section, the logarithms arise naturally when the conformal result is renormalized. Higher powers of logarithms appear when one or more indices are integer and the integrand contains terms of the form $\ln(kz_0)/z_0$.

In all these cases it is possible that the logarithms do not appear to order $\epsilon^{\sigma-3\frac{d}{2}}$. Let us enumerate the special possibilities (n is a positive or vanishing integer):

1. $\sigma = d/2 + 2n$. Logarithms appear at order $\epsilon^{\sigma-3d/2}$.
2. $\sigma = d/2 + 2\nu_i + 2n$. Logarithms multiplying $k_i^{2\nu_i}$ appear at order $\epsilon^{\sigma-3d/2}$.
3. $\sigma = d/2+2\nu_i+2\nu_j$. This case has very special features and we discuss it below.
4. ν_i integer and $\sigma \geq d/2 + 2\nu_i$. Logarithms multiplying $k_i^{2\nu_i}$ appear at order $\epsilon^{2\nu_i+2n-d}$, $n = 0, 1, \ldots, [\frac{1}{2}(\sigma - 3d/2)]$.

Combinations of these possibilities give rise to higher powers of logarithms.

When $\sigma = d/2 + 2\nu_i + 2\nu_j$, the conformal dimension of one of the fields equals the sum of the remaining dimensions. In this "extremal" case, a $\ln\epsilon$ appears in the leading completely non-local term. Therefore, the rescaled correlation function for the CFT diverges logarithmically as $\epsilon \to 0$. The divergence arises from the region near the boundary point where the field with highest dimension is inserted. It seems again that the coupling should vanish in a consistent gravity theory with an ($\epsilon = 0$) CFT dual. However, it is known that the extremal correlators of chiral primary operators in $\mathcal{N} = 4$ SYM have a non-vanishing finite expression. The solution to this problem was given in [51]: the supergravity theory contains couplings with different number of derivatives, such that the divergence and all the bulk contributions cancel and only surface terms remain.[4] In fact, the very same mechanism takes place for two-point functions, which are the simplest example of extremal correlators. In that

[4]Usually, one performs field redefinitions to get rid of the higher derivative couplings. It turns out that the resulting extremal couplings vanish, but one must include surface terms when performing the field redefinitions [52]. Alternatively, one can continue analitically the conformal dimensions; then the pole in the integral in the region near the boundary is cancelled by a zero in the coupling. Again, a finite result is obtained [51, 53].

case, the contributions of the mass and the kinetic terms are also separately divergent, but they cancel out leaving only the surface contribution (3.14).[5] The correlation functions of extremal n-point functions have been well studied both in supergravity theories and in the dual superconformal theories [51, 54, 55, 56]. It has been shown that the field theory extremal correlators obey non-renormalization theorems, in the sense that their value is independent of the coupling. This is related to the fact that extremal correlators factorize into a product of two-point functions. It has also been shown that next-to-extremal functions are neither renormalized [55, 56, 61] and, more generally, that "near-extremal" functions satisfy special factorization properties [53]. This strongly suggests a generalized consistent truncation for type-IIB supergravity on $AdS_5 \times S^5$ [53], that agrees with explicit calculations of cubic [57, 58, 59] and quartic [60] couplings. These arguments have also been used to conjecture generalized consistent truncation in type-IIB supergravity on $AdS_{(4|7)} \times S^{(7|4)}$, for which the CFTs are not so well known [62]. This agrees with the cubic couplings in [59, 63, 64]. It is plausible that the gravity theory relevant to Randall-Sundrum also has these properties. In the following we assume that this is the case at least for extremal functions (which would otherwise either vanish or diverge for $\epsilon \to 0$).

Let us come back to the finite ϵ AdS/CFT. From the assumption above, our Randall-Sundrum gravity theory contains cubic couplings with different number of derivatives that conspire to render the completely non-local part of extremal three-point functions finite. The extremal three-point functions then reduce to surface terms and can be calculated exactly. As an example, we consider type-IIB supergravity and the correlator studied in [51], $\langle \varphi_{\Delta_1} \varphi_{\Delta_2} s_{\Delta_1+\Delta_2} \rangle^{1\text{PI}}$. φ and s are boundary values of Kaluza-Klein modes (from the S^5 reduction) of the dilaton and of a mixture of the 4-form and the graviton (with indices on the sphere). Integrating by parts and using of the equations of motion, this correlator is reduced to the surface term

$$\epsilon^{-d+3} \lim_{z_0 \to \epsilon} \left[\partial_0 \mathcal{K}_{\nu_1}(z_0, \vec{k_1}) \partial_0 \mathcal{K}_{\nu_2}(z_0, \vec{k_2}) \partial_0 \mathcal{K}_{\nu_1+\nu_2+d/2}(z_0, \vec{k_3}) \right]. \tag{3.41}$$

The leading non-local part was found in [51] to have the factorized structure

$$\epsilon^{2(\nu_1+\nu_2)-3d/2} \left[k_1^{2\nu_1} \ln(k_1 \epsilon) \right] \left[k_2^{2\nu_2} \ln(k_2 \epsilon) \right]. \tag{3.42}$$

The logarithms only appear for integer ν_i. This is the Fourier transform of the product of two (renormalized) two-point functions. Without further effort we can

[5] As in any ∞-∞ situation one has to regularize to find a definite answer. D.Z. Freedman et al. used the cutoff regularization and found a result that agrees with the Ward identity relating the two-point function of two scalars and the three-point function of two scalars and a current [45]. Since the latter is power-counting finite for non-coincident points, it follows that the normalization obtained in [45] is universal for any regularization preserving the gauge symmetry for non-coincident points. Analytical continuation also respects the gauge symmetries but it cannot be used for the particular case of the two-point function (since one cannot avoid going into a superextremal situation).

compute the local and semilocal terms in (3.41) as well, and see whether this factorization property holds for the complete regularized functions. We find that it does: the regularized extremal three-point function is, to order $\epsilon^{2(\nu_1+\nu_2-3d/2)}$, a product of two-point functions:

$$\left\langle \varphi_{\nu_1}(\vec{k}_1)\varphi_{\nu_2}(\vec{k}_2)s_{\nu_1+\nu_2+d/2}(\vec{k}_3)\right\rangle^{1\mathrm{PI}} \propto \langle\varphi_{\nu_1}(\vec{k}_1)\varphi_{\nu_1}(-\vec{k}_1)\rangle^{1\mathrm{PI}} \times \times\langle\varphi_{\nu_2}(\vec{k}_2)\varphi_{\nu_2}(-\vec{k}_2)\rangle^{1\mathrm{PI}} . \tag{3.43}$$

It is reasonable to expect that the factorization of regularized extremal correlation functions into products of two-point functions holds to this order in any gravity theory with generalized consistent truncation. On the other hand, as we mentioned above, we have found that when some of the fields have dimension $d/2$, the generalization of the prescription (3.25) does not lead to a correct non-local part of the three-point correlator (at least in the extremal case). This deserves further study.

4. The holographic description of RS2

4.1 Regularization and renormalization of the conformal theory

Any CFT has vanishing beta functions, i.e. contains no divergencies. However, in the AdS/CFT correspondence one does not consider the pure CFT but its deformations by composite operators built out of the elementary field of the theory. The insertions of operators introduce UV divergencies that must be cancelled by local counterterms. Consider two-point correlation functions in a d-dimensional CFT,

$$\langle\mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle = \left[\frac{\delta W^{\mathrm{CFT}}[\varphi]}{\delta\varphi_\Delta(x)\delta\varphi_\Delta(0)}\right]_{\varphi=0} \propto \frac{1}{x^{2\Delta}} . \tag{4.1}$$

Since in this section all the vectors are d dimensional, we use x to indicate both a vector and its euclidean modulus. Δ is the conformal dimension of the operator and φ_Δ is the source coupled to the operator in the path integral (2.3). For $\Delta \geq d/2$ this expression is too singular at coincident points ($x = 0$) to behave as a tempered distribution. In other words, its Fourier transform is divergent. To make sense of it one must renormalize the deformed CFT. The counterterms one needs for this function have the form

$$S^{\mathrm{CFT\,ct}} \supset \int d^d x \varphi_\Delta(x) Q_{\Delta,\Delta}\varphi_\Delta(x) , \tag{4.2}$$

where $Q_{\Delta,\Delta} = c_0 + c_1\Box + \cdots$ is a differential operator of mass dimension 2Δ. We can for instance use dimensional regularization and minimal substraction to find the renormalized expression [65, 66]

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle^R \propto \Box^{[\Delta-d/2]+1}\frac{1}{x^{2(\Delta-[\Delta-d/2]-1)}} \tag{4.3}$$

for non-integer $\Delta - d/2$ and

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle^R \propto \Box^{\Delta-d/2+1}\frac{\ln(xM)}{x^{d-2}} \tag{4.4}$$

for integer $\Delta - d/2$. In latter case one cannot avoid introducing a dimensionful scale M. Since dimensional regularization only detects logarithmic divergencies, the counterterm operator is simply $Q_{2\Delta} = 0$ for non-integer $\Delta - d/2$, and $Q_{2\Delta} \propto \frac{1}{\varepsilon}\Box^{\Delta-d/2}$ for integer $\Delta - d/2$. In both renormalized expressions, the derivatives are prescribed to act by parts on test functions, as in differential renormalization [67]. A change of renormalization scheme would only modify the renormalized result by local terms.

Three-point functions have both subdivergencies and overall divergencies. The former can be cancelled by a local counterterm with two φ coupled to one CFT operator, and the latter by a local counterterm cubic in φ. The local counterterms in the action generate non-local counterterms for the generating functional when the CFT are integrated out. Since the pure CFT is finite, we only need the counterterms that correct the singular behaviour when external points coincide. That is to say, we need not worry about internal points in a Feynman diagram description of the correlators. If we wanted to calculate correlation functions involving both operators and elementary fields of the CFT, A, we would also need counterterms mixing φ and A. As far as we know, the renormalization of the CFT in the context of AdS/CFT has only been studied in [40, 42]. We believe that this is an important aspect of the correspondence that deserves further study. Here we are mainly interested in the CFT at the regularized level. Different regularizations will be dual to different regularizations of the AdS theory and we are interested in the regularization relevant for Randall-Sundrum. The only thing we know about this regularization is that the regulator is a dimensionful parameter (which excludes dimensional regularization). Nevertheless, many of the features we found in AdS can be shown on general grounds to have its counterpart in the regularized CFT.

It will be convenient to use differential regularization [67].[6] This method is well suited for conformal theories as it naturally works in coordinate space and only modifies correlation functions at coincident points [68, 69, 70]. The idea of differential regularization and renormalization is simple: substitute expressions that are too singular at coincident points by derivatives of well-behaved distributions, such that the original and the modified expression are identical at non-coincident

[6]This method can be understood either as a regularization or as a renormalization procedure, and we shall use the terms "differential regularization" and "differential renormalization" to distinguish both interpretations.

points. The derivatives are prescribed to act by parts on test functions, disregarding divergent surface terms around the singularity. For instance, for $d = 4$,

$$\frac{1}{x^6} \longrightarrow \frac{-1}{32} \Box\Box \frac{\ln(xM)}{x^2} + a_0 \mu^2 \delta(x) + a_1 \Box \delta(x) \,. \tag{4.5}$$

In a Fourier transform the total derivatives just give powers of momenta. Since both sides of (4.5) must only be equal for non-zero x, the local terms are arbitrary. Note that changing the mass scale M, which is required to make the argument of the logarithm dimensionless, is equivalent to a redefinition of a_1. This is related to renormalization group invariance. Although the r.h.s. of (4.5) is a correct renormalized expression, here we want to interpret the r.h.s. of (4.5) as the regularized value of the l.h.s. Since on the AdS side there is one single dimensionful parameter, we take $M = \mu = 1/\epsilon$ and consider $\epsilon k \ll 1$ for any momentum k in the Fourier transformed functions. Note that in an arbitrary regularization we could add more local and non-local terms that vanish as $\epsilon \to 0$. They would correspond to higher order terms that do not appear in the approximation we used in the AdS calculations. Some useful differential identities and Fourier transforms are collected in appendix A.

4.2 Two-point functions

The two-point function of two scalar operators of dimension Δ is determined, for non-coincident points, by conformal invariance:

$$\langle \mathcal{O}_\Delta(x) \mathcal{O}_\Delta(0) \rangle = N \frac{1}{x^{2\Delta}} \,, \qquad x \neq 0 \,. \tag{4.6}$$

N is a constant depending on the normalization of the operators. We shall fix it below such that it agrees with the AdS result. In order to extend this expression to a well-defined distribution over all space, we use differential regularization. Again, we have to distinguish two cases: integer $\nu = \Delta - d/2$ and non-integer ν. For non-integer ν the regularized expression is

$$\begin{aligned} \langle \mathcal{O}_\Delta(x) \mathcal{O}_\Delta(0) \rangle = N \Bigg[& C_\Delta \Box^{[\nu]+1} \frac{1}{x^{2(\Delta - [\nu] - 1)}} + a_0 \epsilon^{-2\nu} \delta(x) + \\ & + a_1 \epsilon^{-2(\nu-1)} \Box \delta(x) + \cdots + a_{[\nu]} \epsilon^{-2(\nu - [\nu])} \Box^{[\nu]} \Delta(x) \Bigg] \end{aligned} \tag{4.7}$$

with

$$C_\Delta = \frac{\Gamma(\Delta - [\nu] - 1) \Gamma(\nu - [\nu])}{4^{[\nu]+1} \Gamma(\Delta) \Gamma(\nu + 1)} \,. \tag{4.8}$$

This is a well-defined distribution when the derivatives act "by parts". Using the relation (3.21), a convenient normalization N and the Fourier transforms in appendix A, we find

$$\begin{aligned} \langle \varphi_\nu(k) \varphi_\nu(-k) \rangle^{\text{1PI}} = L^{-1} \left(\frac{L}{\epsilon} \right)^d \Bigg[& \bar{a}_0 - \bar{a}_1 k^2 \epsilon^2 + \cdots + (-1)^{[\nu]} \bar{a}_{[\nu]} k^{2[\nu]} \epsilon^{2[\nu]} + \\ & + \frac{2\Gamma(1-\nu)}{4^\nu \Gamma(\nu)} k^{2\nu} \epsilon^{2\nu} \Bigg] \,, \end{aligned} \tag{4.9}$$

which agrees with the structure in (3.20) to order $\epsilon^{2\nu-d}$. We have defined

$$\bar{a}_i = -\frac{2\nu\Gamma(\Delta)}{\pi^{d/2}\Gamma(\nu)} a_i \,. \tag{4.10}$$

Exact agreement is found only for a particular regularization, i.e. for particular values of a_i:

$$\begin{aligned} \bar{a}_0 &= \Delta \,, \\ \bar{a}_1 &= \frac{1}{2(1-\nu)} \,, \\ &\dots \end{aligned} \tag{4.11}$$

For integer ν we find

$$\begin{aligned} \langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle = N \Bigg[& C'_\Delta \Box^{\nu+1} \frac{\ln(x/\epsilon)}{x^{d-2}} + a_0 \epsilon^{-2\nu}\delta(x) + \\ & + a_1 \epsilon^{-2(\nu-1)} \Box\delta(x) + \dots + a_\nu \Box^\nu \delta(x) \Bigg] \end{aligned} \tag{4.12}$$

with

$$C'_\Delta = \frac{\Gamma(d/2)}{4^\nu (2-d)\Gamma(\Delta)\Gamma(\nu+1)} \,. \tag{4.13}$$

The logarithm appears when writing the unregularized expression as a total derivative (see appendix A). Again, this agrees with the AdS calculation for particular values of a_i.

To obtain renormalized two-point functions, one would need counterterms of the form (4.2). In a minimal substraction scheme,

$$\begin{aligned} Q_{2\Delta} = -N \Bigg(& a_0 \epsilon^{-2\nu} + a_1 \epsilon^{-2(\nu-1)} \Box + \dots + a_{[\nu]} \epsilon^{-2(\nu-[\nu])} \Box^{[\nu]} + \\ & + C'_\Delta \frac{4\pi^{d/2}}{\Gamma(d/2-1)} \ln(M\epsilon) \Box^{\nu+1} \Bigg) \,, \end{aligned} \tag{4.14}$$

where the last term only appears if ν is integer, and M is the renormalization scale. The renormalized two-point function in this scheme reads

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle_R = N C_\Delta \Box^{[\nu]+1} \frac{1}{x^{2(\Delta-[\nu]-1)}} \tag{4.15}$$

and

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle_R = N C'_\Delta \Box^{\nu+1} \frac{\ln(xM)}{x^{d-2}} \,, \tag{4.16}$$

for non-integer and integer ν, respectively.

4.3 Three-point functions

The structure of regularized three-point functions is much richer. The reason is that divergencies can appear when the three points are coincident (overall divergence) or when two points coincide and the other is kept apart (subdivergence). For non-coincident points, the correlation functions of three scalar operators are completely determined by conformal invariance (up to normalization):

$$\begin{aligned}\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle &= N\frac{1}{x_{13}^{\Delta_1+\Delta_3-\Delta_2}x_{23}^{\Delta_2+\Delta_3-\Delta_1}x_{12}^{\Delta_1+\Delta_2-\Delta_3}} \\ &= N\frac{1}{x^{\omega_2}y^{\omega_1}(x-y)^{\omega_3}}\,. \end{aligned} \tag{4.17}$$

In the second identity we have changed variables to $x = x_{13}$, $y = x_{23}$, $z = x_3$ (z does not appear due to translation invariance) and defined ω_i as linear combinations of the conformal dimensions. In general, this function can be regularized with differential regularization using the systematic procedure developed in [71]. Here we shall just describe some features of the regularized functions and compare with the corresponding AdS correlation functions. A particularly simple example is worked out in detail in appendix A.

First, we observe that semilocal terms appear in the CFT description when subdivergencies are regularized. There are subdivergencies when, for some i, $\omega_i \geq d$. Then, the resulting semilocal term in momentum space depends only on k_i. This agrees with the AdS calculations: $\beta^{(i)}$ terms are relevant whenever $\sigma \geq d/2 + 2\nu_i$.

Next, let us consider the special situations. Logarithms can arise from either the subdivergencies or the overall divergence.

1. If $\sum_i \omega_i = \sum_i \Delta_i = 2d + 2n$ there is an overall divergence of degree $2n$. This leads to a simple logarithm at order ϵ^0 when there are no subdivergencies, and to higher powers of logarithms if there are subdivergencies.

2. If $\omega_i - d = 2n$, with $n \geq 0$ integer, the regularized function contains a term with a logarithm at order ϵ^0.

3. $\omega_i = 0$ is the extremal case. We discuss it below.

4. If $\omega_i+\omega_j = 2\Delta_k = d+2n$ ($k \neq i,j$) and $w_k \geq d$, the delta functions appearing in the regularization of $x_{ij}^{-\omega_k}$ multiply $x_{ik}^{-2\Delta_k}$, which has degree of divergence $2n$. Therefore, the completely regularized function contains logarithms at orders $\epsilon^{2m-\omega+d}$ with $m = 0, 1, \ldots, [\frac{1}{2}(\omega_k - d)]$.

These different possibilities perfectly agree, after the global rescaling, with the respective ones analyzed in AdS. (We have used the same numbers for each possibility in both descriptions.) Using the decomposition

$$\Box\frac{\ln(x/\epsilon)}{x^2} = \Box\frac{\ln(xM)}{x^2} + 4\pi^2\ln(M\epsilon)\delta(x)\,, \tag{4.18}$$

we find the same powers of logarithms, with the same factors $k_i^{2\nu_i}$. This equation is the CFT counterpart of the splitting (3.40) in AdS. $\Box\frac{\ln(xM)}{x^2}$ should be understood as a renormalized (sub)expression, with M the renormalization scale.

The extremal case occurs when one ω_i vanishes. Then the unregularized three-point function reduces to a product of unregularized two-point functions. Suppose $\omega_3 = 0$. Then,

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle = N\frac{1}{x_{13}^{2\Delta_1}}\frac{1}{x_{23}^{2\Delta_2}}\,. \tag{4.19}$$

This agrees with the structure of the non-local part of the AdS results. Since bringing x_1 and x_2 together does not lead to further divergencies, we only need to regularize $1/x_{13}^{2\Delta_1}$ and $1/x_{23}^{2\Delta_2}$ independently. From the AdS calculation we know that the regularization of each of these functions coincides with the regularization of the two point functions $\langle\mathcal{O}_{\Delta_1}\mathcal{O}_{\Delta_1}\rangle$ and $\langle\mathcal{O}_{\Delta_2}\mathcal{O}_{\Delta_2}\rangle$, respectively. This does not occur in general for subexpressions of non-extremal functions.

To end this section, let us consider briefly the renormalization of the three-point functions. We do not need this for Randall-Sundrum but it is relevant for the (renormalized) AdS/CFT correspondence. It is clear that the overall divergence can be cancelled by a local counterterm trilinear in the fields φ, similar to (4.2). The semilocal divergent terms, on the other hand, cannot be cancelled by local counterterms made out of fields φ only. What we need are counterterms that couple two fields φ to operators of the CFT. To see this, observe that the singular behaviour of the three-point function when, say, $x_1 \sim x_2$ and x_3 is kept appart, is given by the terms with operators of dimension Δ_3 in the OPE of $\mathcal{O}_{\Delta_1}(x_1)$ and $\mathcal{O}_{\Delta_2}(x_2)$:

$$\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2) \sim \cdots + \left[\frac{1}{x_{12}^{\omega_3}}\right]_\epsilon O'_{\Delta_3}(x_1) + \cdots\,, \tag{4.20}$$

where $[\cdots]_\epsilon$ indicates regularization and $\mathcal{O}'_{\Delta_3}$ denotes possible operators of dimension Δ_3, which can be different from the operators dual to φ_{Δ_3} (typically they are double trace operators). Agreement with AdS calculations implies that the regularization of $(1/x)^\omega$ is not universal, i.e. it depends not only on ω but also on the particular operators that appear in the OPE. For extremal correlators, nevertheless, we have seen that the regularization of each factor is identical to the one in the two-point functions. Since the relevant term in the OPE of both extremal three-point functions (for the two lowest-dimensional operators) and two-point functions is the most singular one, this suggests that the regularization of the most singular term of different OPEs (for fixed ω) is universal. The counterterms that cancel the semilocal divergencies are, in a minimal substraction scheme,

$$\begin{aligned} S^{\mathrm{CFT\,ct}} = &-\sum_{i,j}\int d^dx d^dy \varphi_i(x)\left(\sum_k c_{ij}^k\mathcal{O}'_k(x)\,\mathrm{div}\left[\frac{1}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}}\right]_\epsilon\right)\varphi_j(y) + \\ &+\cdots\,, \end{aligned} \tag{4.21}$$

where c_{ij}^k are the Wilson coefficients in the OPE and div denotes the part that diverges when $\epsilon \to 0$. Since the latter is local, $S^{\mathrm{CFT\,ct}}$ is a local functional. (4.21) generalizes (4.2).

5. Holographic description of RS1

The so-called TeV brane of the RS1 scenario acts as a boundary for the AdS geometry at large z_0. By the UV/IR correspondence, this must correspond to some modification of the CFT such that conformal invariance is broken in the IR. The scale below which the deviation from conformal symmetry is significant is given by the inverse of the position of the TeV brane, $z_0 = \rho$. The holographic dual of RS1 has been studied recently in two interesting papers. In the first one [35], N. Arkani-Hamed, M. Porrati and L. Randall proposed some general features of the holographic theory and use them to explain several phenomenological aspects of the model. In the second one [36], R. Rattazzi and A. Zaffaroni stablished important details of the duality using the AdS/CFT correspondence and discussed some phenomenological issues, like flavour symmetries. In this section we simply study correlations functions of the induced fields in the presence of the TeV brane.

The first question Rattazzi and Zaffaroni addressed was whether conformal invariance is broken explicitly or spontaneously. An explicit breaking by a relevant deformation would affect, to a certain extent, the UV behaviour of correlators. In particular, the trace of the stress-energy momentum would be modified. Rattazzi and Zaffaroni showed that in the original model without stabilization the trace is unchanged, so that the breaking is spontaneous. This agrees with the discussion in [46], according to which a change in the boundary conditions at large z_0 corresponds to a different vacuum in the CFT. Moreover, Rattazzi and Zaffaroni found the corresponding Goldstone pole (associated to the radion, which is a modulus) using both the effective lagrangian of the radion and the AdS/CFT rules (what we called the D approach). So, conformal symmetry is non linearly realized in the original RS1 (and further broken by the UV cutoff). It was also checked in [36] that the deviation from conformality of the non-local part of (d dimensional) massless two-point functions is exponentially suppressed for distances $x \ll \rho$. This indicates that the operator adquiring a vev has formally infinite conformal dimension. The Golberger-Wise stabilization mechanism, on the other hand, introduces an explicit breaking that in the holographic picture corresponds to a deformation by an almost marginal deformation. This explicit breaking gives a small mass to the radion. In order to generate a hierarchy, it is important that the deformation be almost marginal. In the following we consider the model without stabilizing scalars and refer to [35, 36] for the interesting holographic interpretation of the Golberger-Wise mechanism.

In RS1, the TeV brane does not alter the background metric in the region between the two branes. But it does change the large z_0 boundary conditions of the fields propagating in this background: since ρ is an orbifold fixed point, fields must obey a Neumann boundary condition at that point. Then, the propagator with Dirichlet conditions at ϵ picks up a contribution from I_ν Bessel functions:

$$\mathcal{K}(z_0,\vec{k}) = \frac{h(z_0)}{h(\epsilon)} \tag{5.1}$$

with

$$\begin{aligned} h(z_0) = z_0^{d/2}\Big\{ &[2k\rho I_{\nu-1}(k\rho) + (d-2\nu)I_\nu(k\rho)]\ K_\nu(kz_0) + \\ &+ [2k\rho K_{\nu-1}(k\rho) + (2\nu-d)K_\nu(k\rho)]\ I_\nu(kz_0)\Big\}\,. \end{aligned} \tag{5.2}$$

Henceforth we take $L=1$. The 1PI two-point function is obtained by inserting this propagator in the quadratic action, which again reduces to the surface term:

$$\langle \varphi(\vec{k})\varphi(\vec{k})\rangle^{\mathrm{1PI}} = -\epsilon^{1-d} \lim_{z_0\to\epsilon} \partial_{z_0}\mathcal{K}(z_0,\vec{k})\,. \tag{5.3}$$

For both points on the Planck brane and no boundary masses, its Minkowski version is the inverse of the (Neunman) propagator calculated in [72, 73], as expected. The explicit expression can be found in these references. The Kaluza-Klein mass spectrum is now discrete and can be obtained from the poles of this propagator [72]. Only massless bulk fields induce a zero mode on the brane. Negative values of the squared bulk mass induce a single tachyon on the brane. The couplings of the Kaluza-Klein modes at the brane are given by the residues of the corresponding poles [35].

The expansion of (5.3) around $\epsilon = 0$ is more intrincate than in the RS2 model. Instead of looking for a generic expression, we have used Mathematica to calculate the divergent (as $\epsilon \to 0$) local terms and the first non-local term for several values of ν. In all cases we have found exactly the same divergent local terms as in the RS2 model. Moreover, the first non-local term is only modified at scales $k > 1/\rho$ by exponentially small corrections. For integer ν, the logarithm is the same as in the conformal case, but there are small corrections to the local part at order $\epsilon^{2\nu-d}$. This is related to the fact that the conformal anomaly is not altered by the TeV brane. We have also checked numerically that the complete two-point function is virtually independent of ρ in the region $k \gg 1/\rho$ as long as $\rho - \epsilon$ is not very small.

The fact that divergent terms (after rescaling) are not affected by the second brane is actually a particular case of a more general principle: the divergent local part of the effective action in an asymptotically AdS space does not depend on perturbations at large z_0 [16]. It also agrees with the interpretation of RS1 as a CFT in a non-conformal vacuum: the overall UV divergencies are insensitive to the spontanous breaking, which only affects the IR behaviour.

One could also compute three-point functions as in the RS2 model. The local terms should be the same as in RS2. The semilocal terms, on the other hand, will contain some dependence on ρ.

6. Localized fields

In many phenomenologically interesting models, all or part of the Standard model fields are constrained to live on either the Planck or the TeV brane. This is the case of the original Randall-Sundrum proposals, in which only gravity propagates in the bulk. So it is important to incorporate these localized fields in the effective descriptions.[7]

The simplest possibility is having one field ψ localized on the Planck brane in either a RS1 or RS2 model. In general, ψ will have some selfcouplings (including quadratic terms) and couplings to the field ϕ restricted to the Planck brane. The effective action of the whole system is simply the one obtained in the previous sections plus the d-dimensional action involving ψ and its couplings to φ. For example, if the $(d+1)$-dimensional action contains the term

$$S \supset \int_\epsilon d^{d+1}z \sqrt{g(z)} \psi(\vec{z})\psi(\vec{z})\phi(z)\delta(z_0 - \epsilon)\,, \tag{6.1}$$

the effective action will contain a local coupling $\int d^d z \sqrt{g(\epsilon)}\psi(\vec{z})\psi(\vec{z})\varphi(\vec{z})$. In particular, the exchange of φ (with the propagator obtained above) will contribute to the process $\psi\psi \to \psi\psi$. It is also possible that ϕ itself has interactions localized on the Planck brane. Again, these interactions should be simply added to the effective action. One example is the case of boundary mass terms discussed above. What about the holographic description? Since the CFT only represents the bulk degrees of freedom, the fields localized on the Planck brane cannot be part of the CFT. Rather, they couple to it indirectly through φ.

The situation for fields localized at some $z_0 > \epsilon$ is more involved. We shall only find the effective theory in one particular case and make a few qualitative comments about the holographic interpretation. Consider in RS2 a coupling to an external source located at $z_0 = \rho$,

$$S \supset - \int_\epsilon d^{d+1}z \sqrt{g} j \phi(z)\delta(z_0 - \rho)\,. \tag{6.2}$$

This is relevant to Lykken-Randall scenarios, in which the Standard Model fields live on a probe three-brane of infinitesimal tension that does not alter the AdS

[7] We use the term "localized" to refer to fields that do not propagate in the extra dimension. They should not be confused with Kaluza-Klein modes with bounded wave functions, such as the graviton zero mode.

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background. At the quadratic level, the bulk equation of motion in momentum space is

$$z_0^{d+1}\partial_0\left(z_0^{-d+1}\partial_0\phi(z_0,\vec{k})\right)-\left(k^2z_0^2+m^2\right)\phi(z_0,\vec{k})=j(\vec{k})\delta(z_0-\rho)\,. \tag{6.3}$$

The general solution to this equation can be written as

$$\phi(z_0,\vec{k})=\phi^0(z_0,\vec{k})+\sqrt{g(\rho)}j(\vec{k})\Delta_N(\rho;z_0,\vec{k})\,, \tag{6.4}$$

where ϕ^0 is a solution to the homogenous equation and the Neumann propagator Δ_N was defined above. The boundary conditions $\phi(\epsilon,\vec{k})=\varphi(\vec{k})$, $\lim_{z_0\to\infty}\phi(z_0,\vec{k})=0$ fix

$$\phi^0(z_0,\vec{k})=\left(\varphi(\vec{k})-\sqrt{g(\rho)}j(\vec{k})\Delta_N(\rho;\epsilon,\vec{k})\right)\mathcal{K}(z_0,\vec{k})\,. \tag{6.5}$$

The action evaluated on this solution reduces to the surface term

$$\begin{aligned}S^{\rm eff}=\int d^dxd^dy\,\Bigg[&\frac{1}{2}\varphi(\vec{x})\left(-\epsilon^{-d+1}\frac{\partial}{\partial\epsilon}\mathcal{K}(\vec{x};\epsilon,\vec{y})\right)\varphi(\vec{y})+\varphi(\vec{x})\rho^{-d-1}\mathcal{K}(\vec{x};\rho,\vec{y})j(\vec{y})+\\&+\frac{1}{2}j(\vec{x})\rho^{-2(d+1)}\left(\int d^dz\Delta_N(\rho,\vec{x};\epsilon,\vec{z})\mathcal{K}(\vec{z};\rho,\vec{y})-\right.\\&\left.-\Delta_N(\rho,\vec{x};\rho,\vec{y})\right)j(\vec{y})\Bigg]\,.\end{aligned} \tag{6.6}$$

We have used the identity (3.8). We see that the kinetic term is not affected by the presence of the source, but there are additional terms proportional to φj and j^2. From the equations of motion of this action we find the on-shell brane field

$$\varphi(\vec{k})=\sqrt{g(\rho)}j(\vec{k})\Delta_N(\rho;\epsilon,\vec{k})\,, \tag{6.7}$$

which agrees with the one obtained from the original $(d+1)$-dimensional action. On the other hand, in the complete description one can think of a field ϕ being generated by the source j, propagating in $d+1$ dimensions and being eventually absorbed by j at $z_0=\rho$. The amplitud for this process is $g(\rho)\Delta_N(\rho,\rho)$. This information is also contained in the effective action:

$$\begin{aligned}&\left[\frac{\delta}{\delta j(\vec{x_1})}\frac{\delta}{\delta j(\vec{x_2})}\int\mathcal{D}\varphi e^{-S^{\rm eff}}\right]_{j=0}=\\&=g(\rho)\int d^dy\,d^dz\,\mathcal{K}(\vec{x_1};\rho,\vec{y})\langle\varphi(\vec{y})\varphi(\vec{z})\rangle_{j=0}\mathcal{K}(\vec{z};\rho,\vec{x_2})\\&\qquad-g(\rho)\left(\int d^dy\Delta_N(\rho,\vec{x_1};\epsilon,\vec{y})\mathcal{K}(\vec{y};\rho,\vec{x_2})-\Delta_N(\rho,\vec{x_1};\rho,\vec{x_2})\right)\\&=g(\rho)\Delta_N(\rho,\vec{x_1};\rho,\vec{x_2})\,.\end{aligned} \tag{6.8}$$

In the last identity we have used (3.15) and (3.8). In the holographic description, j should couple to the CFT fields, which are dual to the bulk degrees of freedom.

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However, j changes the equations of motion for ϕ, which indicates that the conformal symmetry of the holographic dual must be broken. In order to gain more intuition about this point, it is useful to study the AdS theory from the point of view of the probe brane.

So far we have been discussing the effective theory for the field at the Planck brane, which is obtained by putting $\phi(z)$ on shell for $z_0 > \epsilon$ and keeping $\varphi(\vec{z}) = \phi(\epsilon, \vec{z})$ off shell. Alternatively, we can find an effective theory for the field at the d-dimensional subspace $z_0 = \rho$. We simply have to use in the action the equations of motion of $\phi(z)$ for $z_0 \neq \rho$, with boundary conditions

$$\begin{aligned} [\partial_0 \phi(z)]_{z_0=\epsilon} &= 0\,, \\ \lim_{z_0\to\infty} \phi(z) &= 0\,, \\ \lim_{z_0\to\rho} \phi(z) &= \varphi_\rho(\vec{z})\,. \end{aligned} \tag{6.9}$$

These boundary conditions imply a discontinuity of $\partial_0\phi(z)$ at $z_0 = \rho$. We define the "bulk-to-probe-brane" propagators $\mathcal{K}^\rho_<$ and $\mathcal{K}^\rho_>$ as solutions to the equations of motion with boundary conditions (in momentum space)

$$\begin{aligned} [\partial_0 \mathcal{K}^\rho_<(z_0, \vec{k})]_{z_0=\epsilon} &= 0\,, \\ \lim_{z_0\to\infty} \mathcal{K}^\rho_>(z_0, \vec{k}) &= 0\,, \\ \mathcal{K}^\rho_<(\rho, \vec{k}) = \mathcal{K}^\rho_>(\rho, \vec{k}) &= 1\,. \end{aligned} \tag{6.10}$$

Then, the on-shell field is $\mathcal{K}^\rho_<\varphi_\rho$ ($\mathcal{K}^\rho_>\varphi_\rho$) for $z_0 < \rho$ ($z_0 > \rho$). $\mathcal{K}^\rho_>$ is given by (3.13) substituting ϵ by ρ, while $\mathcal{K}^\rho_<$ is a combination of the modified Bessel functions K_ν and I_ν that we do not write explicitly. Inserting the on-shell ϕ into the action we find the effective action

$$\begin{aligned} S^{\rm eff}_\rho[\varphi_\rho] &= \frac{1}{2}\int_\epsilon d^{d+1}z\,\sqrt{g(z)}\partial^\mu(\phi(z)\partial_\mu\phi(z)) \\ &= \frac{1}{2}\rho^{-d+1}\int d^dz\,\phi(\rho)\left(\lim_{z_0\to\rho^-}\partial_0\phi(z) - \lim_{z_0\to\rho^+}\partial_0\phi(z)\right) \\ &= \frac{1}{2}\rho^{-d+1}\int d^dx\,d^dy\,\varphi(\vec{x})\Big[\partial_0\mathcal{K}^\rho_<(\vec{x};z_0,\vec{y}) - \partial_0\mathcal{K}^\rho_>(\vec{x};z_0,\vec{y})\Big]_{z_0-\rho}\varphi(\vec{y})\,. \end{aligned} \tag{6.11}$$

The explicit result for the 1PI two-point function of the field φ_ρ in momentum space reads

$$\begin{aligned} &\langle\varphi_\rho(\vec{k})\varphi_\rho(-\vec{k})\rangle^{1PI} = \\ &= \frac{\rho^{-d}\left[(\nu-\frac{d}{2})K_\nu(k\epsilon)+k\epsilon K_{\nu-1}(k\epsilon)\right]}{K_\nu(k\rho)\left[\left((\frac{d}{2}-\nu)I_\nu(k\epsilon)+k\epsilon I_{\nu-1}(k\epsilon)\right)K_\nu(k\rho)+\left((\nu-\frac{d}{2})K_\nu(k\epsilon)+k\epsilon K_{\nu-1}(k\epsilon)\right)I_\nu(k\rho)\right]}\,. \end{aligned} \tag{6.12}$$

Now it is straightforward to add sources on the probe brane: they couple linearly to

φ_ρ, just as sources on the Planck brane couple to φ. The inverse of (6.12) gives the propagator for the field φ_ρ, which is identical to $\Delta_N(\rho;\rho,\vec{k})$ (and agrees with [10] for massless fields). After Wick rotation we find one pole at zero momentum for massless fields and one pole at spacelike momentum for negative squared mass. For positive mass the propagator contains no poles. On the other hand, the imaginary part is positive definite and gives the amplitud at $z_0=\rho$ of the Kaluza-Klein modes. This amplitud is oscillatory and vanishes at discrete values of the Kaluza-Klein mass. Once more, we can expand the two-point function around $\epsilon=0$. We find that the leading terms are completely different from the ones in the expansion of the two-point function on the Planck brane. These terms give the correct leading behaviour of the propagator except in the neighbourhood of the pole. The pole is a non-perturbative effect in ϵ.

The effective theory $S_\rho^{\rm eff}$ is useful to study physics on a probe brane at $z_0=\rho$. Its holographic dual is obtained via the "inner AdS/CFT correpondence" proposed in [75] (see also [21] for an application to quasilocalized gravity scenarios). The idea is to decompose the calculation in (6.11) into two parts. First, we evaluate the action with on-shell $\phi(z)$ only for $z_0<\rho$. The resulting action depends on $\phi(z)$ at $z_0\geq\rho$:

$$S_\rho = \tilde{S}_\rho^{\rm eff}[\varphi_\rho] + \frac{1}{2}\int_\rho d^{d+1}z\,\sqrt{g(z)}\left(\partial^\mu\phi\partial_\mu\phi + M^2\phi^2\right), \tag{6.13}$$

where

$$\tilde{S}_\rho^{\rm eff}[\varphi_\rho] = \frac{1}{2}\rho^{-d+1}\int d^dx\,d^dy\,\varphi_\rho(\vec{x})\left[\partial_0\mathcal{K}_<^\rho(\vec{x};z_0,\vec{y})\right]_{z_0=\rho}\varphi_\rho(\vec{y}) \tag{6.14}$$

is an effective action describing the effect of the bulk degrees of freedom between ϵ and ρ. The second term in (6.13) is simply the action of an RS2 model with Planck brane at ρ. It is therefore dual to the same CFT with regulator ρ, perturbed by operators coupled to φ_ρ. The contribution $\tilde{S}_\rho^{\rm eff}$ arises from integrating out the degrees of freedom of the CFT heavier than $1/\rho$. As we can see in (6.14), the (wilsonian) renormalization group flow of the theory from $1/\epsilon$ to $1/\rho$ is not the simple rescaling one would expect in a conformal theory. A more complicated structure arises due to the Dirichlet boundary condition at $z_0=\rho$, which substitutes the condition of regularity at infinity.[8] A change of boundary condition inside AdS corresponds to a different vev for the dual operator, i.e. to a change of vacuum in the CFT [46]. The situation we are describing is then consistent with the interpretation in [35] of the holographic dual of Lykken-Randall (for the Standar type-IIB correspondence) as arising from Higgsing the original CFT: U(N) $\to$ U($N-1$) $\times$ U(1). If this is

[8] In [75] a conformal flow given by $\mathcal{K}_\nu(\rho)$ is found. The reason is that no condition is imposed at the boundary, and the derivative of the on-shell field is implicitly assumed to be continuous at $z_0=\rho$. We observe that if the Planck brane is removed, one should impose regularity of the field as it approaches the AdS boundary. This implies a discontinuous derivative at $z_0=\rho$.

correct, the RG flow leading to (6.14) is the flow from $1/\epsilon$ to $1/\rho$ of a CFT with a Coulomb branch deformation (in the approximation where the metric is fixed). We have mentioned that the first terms in the small ϵ expansion of the φ_ρ two-point function differ from the ones of the φ two-point function. This indicates that the regulator structure is mixed in a non-trivial way by the RG flow. Note that once we reach ρ, the effective action dual to S_ρ is given by the non-local action $\tilde{S}_\rho^{\text{eff}}$ coupled to a CFT in the conformal phase with operator deformations. If the brane at $z_0 = \rho$ is the TeV brane of RS1, the space terminates at ρ and the whole effective theory at that brane is described by $\tilde{S}_\rho^{\text{eff}}$ alone.

In the holographic interpretation we have proposed, using (6.13) as guiding principle, the correct propagator is reproduced when the source couples directly to φ_ρ, as long as the RG flow gives the bilinear term (6.14) (in a low energy approximation). On the other hand, it has been proposed in [46] (see also the earlier discussion in [11]) that the effective theory on the brane can contain an additional coupling of the source to a CFT operator (but no bililear term in φ_ρ):

$$\int \rho^{2\Delta-d} j\mathcal{O}_\Delta \,. \tag{6.15}$$

Integrating out the remaining CFT modes one finds, schematically,

$$\varphi_\rho\langle\mathcal{O}\mathcal{O}\rangle_\rho\varphi_\rho + \rho^{2\Delta-d}j\langle\mathcal{O}\mathcal{O}\rangle_\rho\varphi_\rho + \rho^{4\Delta-2d}j\langle\mathcal{O}\mathcal{O}\rangle_\rho j\,, \tag{6.16}$$

which is analogous to (6.6) and gives the right qualitative low-energy behaviour of the propagator. However, we cannot reproduce in this way the exact propagator since (6.15) and (6.16) contain no information about ϵ. Furthermore, we have not found a definite AdS counterpart of the term (6.15). So, it seems difficult to reconciliate this interpretation with the regularized AdS/CFT correspondence.

7. Discussion

We have calculated generic two-point and three-point correlation functions of induced fields in Randall-Sundrum models (mainly RS2), and shown explicitly that they can be obtained holographically from a *regularized* CFT. Local and semilocal terms for the brane fields arise from the regularization of the theory. These terms are more important at low energies than the non-local ones arising from the unregulated CFT. Alternatively, Randall-Sundrum models can be holographically described as a *renormalized* CFT coupled to a local theory for the induced fields: $S^{\text{CFT}}[A] + S'[A,\varphi]$. $S'[A,\varphi]$ contains the couplings $\varphi\mathcal{O}$ and more complicated couplings that are related to the IR counterterms of the AdS theory. In particular it contains kinetic terms for φ, which can then be naturally treated as a dynamical field. In fact, this is what one should do, since in the AdS theory the induced fields are

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dynamical and not just external sources. In other words, in the complete theory one has to include both sides of (2.4) inside a path integral over φ. Although in this paper we have followed the "regularization" point of view, the "renormalization" approach is more adequate in some discussions, particularly when the holographic theory is taken as the starting point [76, 77] and AdS is just a calculational tool. Indeed, from the point of view of the quantum field theory it is more natural to say that the world is described by a particular renormalized theory, rather than by a theory regularized in a very particular way. From this viewpoint, the Planck brane should not be thought of as a regulator, but rather as an element one has to add to the AdS description to take into account the effect of S'. One can then do model building modifying S', which will correspond to changing the Planck brane by a more complicated object.

For simplicity we have studied only scalar fields. Most of the results generalize to fields with higher spin. The effective theories for the graviton and gauge fields will have gauge symmetries, since a subgroup of the $(d+1)$-dimensional diffeomorphism and gauge invariance acts on the field induced on the brane. In the usual AdS/CFT correspondence, gauge symmetries in the bulk induce global symmetries in the boundary theory. In the holographic Randall-Sundrum theory, these global symmetries are gauged by the induced fields on the brane. An essential point here is that the regularization (or renormalization) of the CFT respects the Ward identities of these symmetries. The Planck brane provides such a regularization (respecting gauge invariance and supersymmetry, which is a hard task from a field-theoretical approach). On the other hand, knowing that the effective theory has a non-trivial symmetry is very useful because the interacting terms are related to the quadratic ones, which are much simpler to calculate.

It is important to examine the approximations and assumptions we have used in this paper. First, we have been mainly interested in a low-energy expansion. For momenta smaller than the inverse AdS radius, the effect of the regularization is local to a good approximation, and can be interpreted as the addition of a local action S' to the renormalized CFT. Higher order effects can be reproduced by a non-local S', but this looks like a rather artificial construction from the field theory side. We observe that this approximation is related to the way in which the Planck brane modifies the standard AdS/CFT correspondence. So, considering string theory instead of supergravity (or small coupling in the field theory) does not help in going beyond the low-energy approximation.

Second, we have ignored throughout the paper the back-reaction of the scalar fields on the metric. If the back-reaction is small, it will only affect the fluctuations of the metric (the graviton), which can be perturbatively studied with the same formalism. However, for some ranges of parameters the back-reaction changes dramatically the AdS background. This effect can be important for the scalars stabilizing the two-brane models [2]. Moreover, it has been shown in [48] that generic solutions preserving four-dimensional Poincaré have a positive-definite potential in the equivalent quantum-mechanical problem, and therefore there are no tachyons in the effective theory on the boundary. Presumably this result applies also to the effective theory at finite ϵ, showing that the instability found in [47] and here for negative squared mass is just an artifact of the approximation of "inert" scalars. The divergencies of the coupled scalar-gravity system have been calculated in [16] for scalars with $\Delta \leq d$ (corresponding to relevant or marginal deformations), while for $\Delta > d$ one finds uncontrollable divergencies and the formalism of [16] cannot be applied. Correlation functions of active scalars were first studied in [2], and it was found that the usual AdS/CFT prescription could not be used when the scalars are not decoupled. Subsequently, a method was developed in [78] to decouple the scalar fluctuations from the graviton ones. This allowed the authors of this paper to compute the two-point functions of the decoupled scalar in the usual way (see also [79]). It would be interesting to apply this method to brane world models.

Third, in our CFT calculations we have always treated φ as an external field and computed φ correlators ignoring the effect of virtual φ interacting with the CFT fields. Since the CFT fields are dual to the bulk degrees of freedom, including such quantum corrections should correspond to a dynamical modification of the AdS background. One might hope that the full quantum theory $S^{\rm CFT} + S'$ is related somehow to the exact coupled gravity-scalar theory in the presence of the Planck brane, but we have not found convincing arguments supporting this idea. At any rate it is clear that if one starts with the d-dimensional theory $S^{\rm CFT} + S'$, the effect of S' in quantum corrections is important. This issue has been briefly discussed in [35]. There it has been pointed out that, in order to agree with the AdS picture, S' must be such that the theory remains conformal in the IR. To protect the scalars of the CFT from getting a mass through radiative corrections, one needs to impose some symmetry on S', such as supersymmetry, that forbids strongly relevant operators (nearly marginal operators can give rise to a Goldberger-Wise stabilization mechanism). We finish these comments observing that the AdS problems with $\Delta > d$ are related to the fact that the perturbed theory is in this case non renormalizable and requires an infinite set of counterterms.

Our analysis of the holographic duals of RS1 and Lykken-Randall models has been less quantitative and we have left many open questions. A more complete study requires a better understanding of the inner AdS/CFT correspondence and its relation with the RG equations satisfied by the correlation functions.

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A. Differential regularization: some useful identities and a simple example

In this appendix we collect several formulae that are useful in differential regularization. Then, we give an explicit example of the regularization of a three-point function.

Differential identities in d dimensions for $x \neq 0$:

$$\frac{1}{x^\beta} = \frac{1}{(2-\beta)(\beta-d)}\Box\frac{1}{x^{\beta-2}}\,, \qquad \beta > d\,, \tag{A.1}$$

$$\frac{1}{x^d} = \frac{1}{(2-d)}\Box\frac{\ln(xM)}{x^{d-2}}\,, \tag{A.2}$$

$$\Box\frac{1}{x^{d-2}} = -\frac{4\pi^{d/2}}{\Gamma(d/2-1)}\delta(x)\,. \tag{A.3}$$

These identites can be used iteratively to find the regularized value of two-point functions that we have written in section 4. For the example below we also need the following identities (for $x \neq 0$). We only write them in 4 dimensions:

$$\begin{aligned}\frac{\ln(xM)}{x^4} &= -\frac{1}{4}\Box\frac{\ln^2(xM)+\ln(xM)}{x^2}\,,\\ \frac{\ln(xM)}{x^6} &= -\frac{1}{64}\Box\Box\frac{2\ln^2(xM)+5\ln(xM)}{x^2}\,.\end{aligned} \tag{A.4}$$

To find the Fourier transforms of the regularized functions we only need the "integration by parts" rule of differential regularization and the following Fourier transforms:

$$\begin{aligned}\frac{1}{x^{2\alpha}} &\longrightarrow \frac{2^{d-2\alpha}\pi^{d/2}\Gamma(d/2-\alpha)}{\Gamma(\alpha)}\,k^{2\alpha-d}\,, \qquad \alpha < \frac{d}{2}\,,\\ \frac{\ln(xM)}{x^{d-2}} &\longrightarrow \frac{2\pi^{d/2}(2-d)}{\Gamma(d/2)}\frac{\ln(k/\bar{M})}{k^2}\,,\end{aligned} \tag{A.5}$$

with $\bar{M} = \frac{2M}{\gamma_E}e^{\psi(d/2-1)}$, $\gamma_E = 1.781\ldots$ and ψ being the Euler constant and Euler psi function, respectively.

Let us consider now the regularization of the three-point correlator with $\Delta_1 = \Delta_2 = 3$, $\Delta_3 = 4$, in dimension $d = 4$. For non-coincident points,

$$\langle \mathcal{O}_3(x_1)\mathcal{O}_3(x_2)\mathcal{O}_4(x_3)\rangle = N\frac{1}{x^4y^4(x-y)^2}\,, \tag{A.6}$$

where x and y were defined in the text. First, we have to regularize the divergencies at $x \sim 0$ and $y \sim 0$ for $x \neq y$. We find

$$\left(-\frac{1}{2}\Box_x\frac{\ln x/\epsilon}{x^2} + b_0^{(1)}\delta(x)\right)\left(-\frac{1}{2}\Box_y\frac{\ln y/\epsilon}{y^2} + b_0^{(2)}\delta(y)\right)\frac{1}{(x-y)^2}\,. \tag{A.7}$$

Using the Leibniz rule and (A.3) we can write this expression as a total derivative of a well-behaved distribution plus divergent terms depending on just one variable (times deltas):

$$
\begin{aligned}
&\frac{1}{4}\Box_x\Box_y\frac{\ln(x/\epsilon)\ln(y/\epsilon)}{x^2y^2(x-y)^2}-\pi^2\Box_x\left[\frac{\ln^2(x/\epsilon)}{x^4}\delta(x-y)\right]-\\
&\quad-\frac{b_0^{(2)}}{2}\Box_x\frac{\ln(x/\epsilon)}{x^4}\delta(y)-\frac{b_0^{(1)}}{2}\Box_y\frac{\ln(y/\epsilon)}{y^4}\delta(x)+2\pi^2\frac{\ln(x/\epsilon)}{x^6}\delta(x-y)+\\
&\quad+\left(2\pi^2b_0^{(2)}\frac{\ln(x/\epsilon)}{x^2}+b_0^{(1)}b_0^{(2)}\frac{1}{x^2}\right)\delta(x)\delta(y)\,.
\end{aligned}
\tag{A.8}
$$

Finally, we regularize the overall divergence using the differential regularization identities above. The final regularized function reads

$$
\begin{aligned}
\langle\mathcal{O}_3(x_1)\mathcal{O}_3(x_2)\mathcal{O}_4(x_3)\rangle=&\frac{1}{4}\Box_x\Box_y\frac{\ln(x/\epsilon)\ln(y/\epsilon)}{x^2y^2(x-y)^2}+\\
&+\frac{\pi^2}{4}\Box_x(\partial_x+\partial_y)^2\left[\frac{\ln^2(x/\epsilon)+\ln(x/\epsilon)}{x^2}\delta(x-y)\right]+\\
&+\frac{b_0^{(2)}}{8}\Box_x^2\frac{\ln^2(x/\epsilon)+\ln(x/\epsilon)}{x^2}\delta(y)+\\
&+\frac{b_0^{(1)}}{8}\Box_y^2\frac{\ln^2(y/\epsilon)+\ln(y/\epsilon)}{y^2}\delta(x)-\\
&-\frac{\pi^2}{36}(\partial_x+\partial_y)^4\left[\frac{2\ln^2(x/\epsilon)+5\ln(x/\epsilon)}{x^2}\delta(x-y)\right]+\\
&+\left(a_0\frac{1}{\epsilon^2}+a_1^{(1)}\Box_x+a_1^{(2)}\Box_y+a_1^{(3)}(\partial_x+\partial_y)^2\right)[\delta(x)\delta(y)]\,.
\end{aligned}
\tag{A.9}
$$

Observe that ∂_x, ∂_y and $\partial_x+\partial_y$ Fourier transform into $-ik_1$, $-ik_2$ and $-ik_3=-i(k_1+k_2)$, respectively. We could also have obtained a longer expression explicitly symmetric under $x_1\leftrightarrow x_2$.

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